

Modular Invariants and Twisted Equivariant K -theory II: Dynkin diagram symmetries

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December 9, 2010

Abstract

The modular invariant partition functions of conformal field theory (CFT) have a rich interpretation within von Neumann algebras (subfactors), which has led to the development of structures such as the full system (fusion ring of defect lines), nimrep (cylindrical partition function), alpha-induction, etc. Modular categorical interpretations for these have followed. More recently, Freed-Hopkins-Teleman have expressed the Verlinde ring of conformal field theories associated to loop groups as twisted equivariant K -theory. For the generic families of modular invariants (i.e. those associated to Dynkin diagram symmetries), we build on Freed-Hopkins-Teleman to provide a K -theoretic framework for other CFT structures, namely the full system, nimrep, alpha-induction, D-brane charges and charge-groups, etc. We also study conformal embeddings and the E_7 modular invariant of $SU(2)$, as well as some families of finite groups. This new K -theoretic framework allows us to simplify and extend the less transparent, more ad hoc descriptions of these structures obtained within CFT using loop group representation theory.

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1 Introduction

Let G be a compact, connected, simply-connected Lie group and LG its loop group. Its representation ring R_G can be realized as the equivariant (topological) K -group $K_G^0(\text{pt})$ of G acting trivially on a point pt . Similarly, Freed–Hopkins–Teleman [30, 31, 32, 33] identify the Verlinde ring $\text{Ver}_k(G)$ with the twisted equivariant K -group ${}^\tau K_G^{\dim G}(G)$ for some twist $\tau \in H_G^3(G; \mathbb{Z})$ depending on the level k , where G acts on itself by conjugation. $\text{Ver}_k(G)$ can be regarded as an analogue of R_G for loop groups. The ring structure of $\text{Ver}_k(G)$ is recovered from the push-forward of group multiplication $m : G \times G \rightarrow G$, whereas its R_G -module structure comes from the push-forward of the inclusion $1 \hookrightarrow G$ of the identity.

Freed–Hopkins–Teleman interpret their theorem within the context of 3-dimensional topological quantum field theory, namely as an expression as a twisted equivariant K -group of the functorial image of the torus, or equivalently the Grothendieck ring of the image of a circle. But it is also a theorem in 2-dimensional conformally invariant quantum field theory (CFT), namely a K -theoretic expression for the fusion ring, for the CFT describing strings living on G .

A CFT consists of two semi-autonomous *chiral* halves (essentially vertex operator algebras); they are glued together into the *full* CFT by the 1-loop partition function, called the *modular invariant*; knowing this modular invariant is equivalent to knowing

the space which Segal’s functor associates to the circle. The fusion ring is usually regarded as chiral data. But a rich structure (full system, alpha-induction etc) to full CFT was realised mathematically in the subfactor interpretation by Böckenhauer, Evans, Kawahigashi, Pinto etc (see e.g. the reviews [6, 7]), and transported to the categorical setting by Fuchs, Fröhlich, Ostrik, Runkel, Schweigert etc (see e.g. the review [34]).

This CFT interpretation of Freed-Hopkins-Teleman’s theorem suggests several other related developments, many of which are provided in this paper. The different CFTs associated to a given group G and level k are largely parametrised by their modular invariants. For each choice of *viable* or *sufferable* modular invariant (that is, those realised by a CFT), there are a number of structures reminiscent of the Verlinde ring. We seek their K -theoretic descriptions.

When G is a finite group, Evans [21] does this, guided by the braided subfactor approach. The sufferable modular invariants for the twisted double $\mathcal{D}^\tau(G)$ are parametrized by pairs (H, ψ) for a subgroup H of $G \times G$ and $\psi \in H_H^2(\text{pt}; \mathbb{T})$ [25, 58]. Let $H \times H$ act on $G \times G$ on the left and right: $(h_L, h_R) \cdot (g, g') = h_L(g, g')h_R^{-1}$. Then ${}^{\tau, \psi}K_{H \times H}^0(G \times G)$ can be identified with the *full system* (the fusion ring of defect lines), and again ${}^{\tau, \psi}K_{H \times H}^1(G \times G) = 0$ (at least when the twists τ and ψ are trivial). Choosing H to be the diagonal subgroup isomorphic to G recovers the Verlinde ring.

As warm-up, we work out explicitly here the story for certain classes of finite groups (cyclic and dihedral). But our real interest is Lie groups. From this point of view, Lie groups are much more complicated than finite groups — for example, there is no direct analogue of the (H, ψ) parametrization of viable modular invariants — but [22] argues that similar extensions of Freed–Hopkins–Teleman can be expected. This paper establishes several of these, going far beyond [22].

We start with the case of an n -torus, which we work out in complete detail. Its Verlinde ring was determined K -theoretically in [32] (the bundle picture was described in [22]); in section 3 we find the full system, nimrep, alpha-induction etc for any modular invariant. We find that all modular invariants are sufferable, with a unique nimrep.

Focus now on G being compact, connected, and simply-connected. The most important source of modular invariants for these G are the simple-current invariants. These correspond to strings living on the non-simply-connected groups G/Z (for Z a subgroup of the centre of G). Here we now have a complete theory. The full system is given by the twisted K -theory of $G \times G$ acting diagonally on $(G/Z_0) \times (G/Z_0)$ for some subgroup Z_0 of Z we describe. By contrast, ${}^\tau K_G^{\dim G}(G/Z)$ for G acting adjointly on G/Z is the associated nimrep (and again vanish for $\dim(G) + 1$), and ${}^{\tau'} K_{G/Z_0}^{\dim G}(G/Z_0)$ again for the adjoint action gives the neutral system (i.e. describes the maximal chiral extension). We give the alpha-inductions below; D-brane charges are obtained from the natural map ${}^\tau K_G^{\dim G}(G/Z) \rightarrow {}^\tau K^{\dim G}(G/Z)$ forgetting G -equivariance. We explicitly verify this for $G = \text{SU}(2)$ and $Z = \{\pm I\}$.

A nimrep for these simple-current modular invariants had been conjectured in [4, 39]; in Theorem 4 below we prove this conjecture and rewrite the the nimrep in

a considerably simpler way. We were led to this description by trying to match the conjectured nimrep with the expected K -group.

We also give in section 4.2 a complete K -homological description for the case of twists by outer automorphisms of a Lie group G . In section 5.3 we give the K -homological description of outer automorphism twists of the nonsimply-connected groups $SU(n)/Z_d$. For any given G , the simple-current modular invariants and their twists by outer automorphisms comprise the generic modular invariants of G (i.e. for each G we expect only finitely many additional modular invariants). We compare our K -homological descriptions with the analogous descriptions (when they exist) coming from conformal field theory (see e.g. [37, 38, 39, 40]). They match beautifully.

For the most familiar case of $G = SU(2)$, all that remains are the three exceptional modular invariants. Of these, two (namely \mathbb{E}_6 and \mathbb{E}_8) are due to *conformal embeddings*. In [22] we approximated the full system for any conformal embedding, LH at level k in LG at level ℓ , by ${}^\tau K_H^{\dim H}(G)$ where the group H acts adjointly on G . This is clearly a part of the story: we could see the level change in going to the subgroup (e.g. from G_2 level 1 to $SU(2)$ level 28), and we observed that McKay's A-D-E name for the largest finite stabiliser for H -conjugation on G matched the Cappelli-Itzykson-Zuber name for the corresponding modular invariant. In this paper we propose (in the spirit of [21]) to realize the full system by ${}^\tau K_0^{H \times H}(G \times G)$ where now the action is given by $(h, h').(g, g') = (hgh'^{-1}, hg'h'^{-1})$. This tightens the match with the full system. Finally, we can K -theoretically identify most of the full system for the remaining $SU(2)$ modular invariant \mathbb{E}_7 . See section 6 for the details of our treatment of these exceptional modular invariants.

The point of our work is not merely to translate everything in RCFT into K -theory, but rather to demonstrate that the latter provides systematic tools for understanding the former. For example, as explained in section 4.2, the easiest way to prove the conjectured picture in RCFT for nimreps associated to outer automorphisms would be to relate it to our K -homological description. Indeed as already mentioned, we were led to our proof (Theorem 4 below) that the conjectured nimrep for simple-current invariants *is* a nimrep, by K -homological considerations. Or consider a modular invariant \mathcal{Z} . It may or may not correspond to an RCFT; for example in general there will not exist the necessary extra structure, e.g. a nimrep, compatible with it. Subfactor theory elegantly describes these extra structures but there still remains the problem of showing existence. For example, to establish that all modular invariants for $SU(3)$ *are* realised by subfactors, [26, 27] start with the nimrep graphs, construct for them the Boltzmann weights, construct from this the subfactors, compute the nimreps and recover the graphs they started with. What is needed is something more systematic, which for instance does not need to know what the nimrep graphs are expected to be. A motivation for our present and future K -theory work is to help provide tools for showing that certain modular invariants are realised by RCFT and have the extra structure. Here by K -theory we really mean the concepts of vector bundle/Fredholm module/cycle, not just the equivalence classes (i.e. K -groups themselves) which would only produce graphs. This point is also

discussed at the end of [22].

The modular invariant is an integral matrix indexed by the primaries (the preferred basis in the Verlinde ring), and as such is a linear map between KK -groups. As proposed in [22], it should be understood as an element of KK -theory. Likewise for alpha-induction and sigma-restriction. In the special case of the double of finite groups (with trivial H^3 twist), a natural basis can be found in which the modular group acts by permutation matrices, and so in this case these matrices can also be interpreted as KK -elements. Given the accomplishments of this paper, developing these pictures is the natural next step.

2 Review and notation

2.1 Groups, representations and cohomology

See [22] for details and references. For any compact finite-dimensional group G , we write R_G for the representation ring (equivalently, character ring) of the group G , the span over \mathbb{Z} of the (isomorphism classes of) irreducible representations ($=$ *irreps*). For arbitrary compact, connected, simply-connected G of rank r , we write ρ_λ for the irrep with highest weight $\lambda \in \mathbb{Z}_{\geq 0}^r := P_+(G)$. Write $\Lambda_1, \dots, \Lambda_r$ for the fundamental weights; then as is well-known R_G is the polynomial ring $\mathbb{Z}[\rho_{\Lambda_1}, \dots, \rho_{\Lambda_r}]$.

It is convenient to introduce special notation for three common R_G :

$$\begin{aligned} R_{\mathrm{SU}2} &= \mathbb{Z}[\sigma] = \mathrm{span}\{\sigma_1, \sigma_2, \dots\}, \\ R_{\mathrm{O}2} &= \mathbb{Z}[\delta, \kappa] / (\delta\kappa = \kappa, \delta^2 = 1) = \mathrm{span}\{1, \delta, \kappa_1, \kappa_2, \dots\}, \\ R_{\mathbb{T}} &= \mathbb{Z}[a^{\pm 1}] = \mathrm{span}\{1, a, a^{-1}, a^2, a^{-2}, \dots\}, \end{aligned}$$

where σ_i is the i -dimensional $\mathrm{SU}(2)$ -representation (so $\sigma = \sigma_2$ is the defining representation), $\delta = \det$, κ_i is the two-dimensional $\mathrm{O}(2)$ -representation with winding number i (so $\kappa = \kappa_1$ is the defining representation), and a^i is the one-dimensional representation for the circle $\mathrm{SO}(2) = \mathrm{U}(1) = S^1 = \mathbb{T}$ with winding number i .

It is convenient to extend the notation to $\sigma_0 = 0$, $\kappa_0 = 1 + \delta$, and $\sigma_{-n} = -\sigma_n$ and $\kappa_{-n} = \kappa_n$ for all $n \in \mathbb{Z}_{>0}$. Some inductions we need are Dirac induction $\mathrm{D}\text{-}\mathrm{Ind}_T^{\mathrm{SU}2}$, which sends a^i to σ_i , and $\mathrm{Ind}_T^{\mathrm{O}2}$, which sends a^i to κ_i .

When the fundamental group $\pi_1(G)$ is not trivial, we can define *spinors*. Consider $G = \mathrm{SO}(2n+1)$, the quotient of $\mathrm{Spin}(2n+1)$ by its centre $Z = \{1, z\}$: the $\mathrm{Spin}(2n+1)$ -irreps have highest weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$; the value of the λ -irrep on $z \in Z$ is $(-1)^{\lambda_n}$. The representation ring $R_{\mathrm{SO}(2n+1)}$ is the \mathbb{Z} -span of the irreps with λ_n even; the spinors are the irreps with λ_n odd, and their \mathbb{Z} -span $R_{\mathrm{SO}(2n+1)}^-$ is a module for $R_{\mathrm{SO}(2n+1)}$.

Suppose $\phi : H \rightarrow G$ is a group homomorphism. Write Δ_H for the diagonal subgroup $\{(\phi(h), \phi(h)) : h \in H\}$ of $G \times G$. Write ‘ H^{ad} on G ’ for the adjoint action of H on G , where $h.g = \phi(h)g\phi(h)^{-1}$, and write ‘ H^L on G ’ (resp. ‘ H^R on G ’) for the left (right) action $h.g = \phi(h)g$ (resp. $h.g = g\phi(h^{-1})$).

For reasons explained next subsection, we are interested in group cohomology, in particular $H_G^1(X; \mathbb{Z}_2)$ and $H_G^3(X; \mathbb{Z})$. These are defined by $H_G^n(X; A) = H^n((E_G \times X)/G; A)$ where $B_G = E_G/G$ is the classifying space of G . Some useful facts are the *Künneth formula*, which implies

$$H_{G \times H}^n(X; A) \simeq \bigoplus_{m=0}^n H_G^m(X; A) \otimes_{\mathbb{Z}} H_H^{n-m}(X; A) \quad (2.1)$$

and when N is a normal subgroup of G acting freely on X , then we have the natural isomorphism

$$H_G^*(X; A) \simeq H_{G/N}^*(X/N; A). \quad (2.2)$$

It is often convenient to compute these cohomology groups using spectral sequences via the fibration $X \rightarrow (E_G \times X)/G \rightarrow B_G$. Then $E_2^{p,q} = H_G^p(\text{pt}; H^q(X; \mathbb{Z}))$. When G is compact, connected, and simply-connected, $H_G^*(\text{pt}; \mathbb{Z}) \simeq \mathbb{Z} \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{Z} \oplus \dots$ and $H^*(G; \mathbb{Z}) = \mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z} \oplus \dots$, while $H_G^*(\text{pt}; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus \dots$ and $H^*(G; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus 0 \oplus \dots$. For $G = \mathbb{Z}_n^m$, $H_G^*(\text{pt}; \mathbb{Z}) \simeq \mathbb{Z} \oplus 0 \oplus \mathbb{Z}_n^m \oplus 0 \oplus \mathbb{Z}_n^{m(m^2+5)/6} \oplus \dots$ and $H_G^*(\text{pt}; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_{\gcd(n,2)}^m \oplus \dots$, which identifies both $H_{G^{\text{ad}}}^3(G; \mathbb{Z})$ and $H_{G^{\text{ad}}}^3(G\omega; \mathbb{Z})$ with $H_G^0(\text{pt}; H^3(G; \mathbb{Z})) = \mathbb{Z}$. From this we find for example that, for Z a subgroup of the centre of G ,

$$H_{G^{\text{ad}}}^3(G/Z; \mathbb{Z}) \simeq H_{G^{\text{ad}} \times Z^L}^3(G; \mathbb{Z}) \simeq \mathbb{Z}, \quad H_{G^{\text{ad}}}^1(G/Z; \mathbb{Z}_2) \simeq \text{Hom}(Z, \mathbb{Z}_2). \quad (2.3)$$

Given any nontrivial $\epsilon \in H_G^1(\text{pt}; \mathbb{Z}_2)$, i.e. a continuous group homomorphism $\epsilon : G \rightarrow \mathbb{Z}_2$, the subgroup $H := \ker \epsilon$ of G is index 2. There are two types of G -irreps ρ (see [22, Sect.2.1] for details):

type $\frac{2}{1}$: there is an inequivalent G -irrep ρ' such that $\text{Res}_H^G \rho \simeq \text{Res}_H^G \rho'$ is irreducible;
type $\frac{1}{2}$: $\text{Res}_H^G \rho = \rho_1 \oplus \rho_2$ where ρ_i are H -irreps.

By the *graded representation ring* ${}^\epsilon R_G$ [30, Sect.4], [22, Sects.1.2,2.1] we mean the \mathbb{Z} -span of formal differences $\rho_1 \ominus \rho_2 = -(\rho_2 \ominus \rho_1)$ over all *type* $\frac{1}{2}$ G -irreps ρ . By ${}^\epsilon R_G^1$ we mean the \mathbb{Z} -span of formal anti-symmetrisations $\rho^- := (\rho - \rho')/2 = -\rho'^-$ over all *type* $\frac{2}{1}$ G -irreps ρ .

2.2 Twisted equivariant K -theory

We are interested in K -groups ${}^\tau K_G^*(X)$ and K -homology ${}^\tau K_*^G(X)$, where X is a nice space with a nice G -action, and the twist τ lies in $H_G^1(X; \mathbb{Z}_2) \times H_G^3(X; \mathbb{Z})$. Here, ‘ \times ’ denotes Cartesian product; the group structure is more complicated, involving the Bockstein homomorphism, see e.g. [22, eqn.(1.4)]. For the definition, basic properties, and original references of twisted equivariant K -theory, see [1, 50, 22].

Bott periodicity says ${}^\tau K_G^{i+2}(X) = {}^\tau K_G^i(X)$ and ${}^\tau K_{i+2}^G(X) = {}^\tau K_i^G(X)$, for all G, X, i . The additive groups ${}^\tau K_G^*(X)$ and ${}^\tau K_*^G(X)$ carry an R_G -module structure. When $\epsilon \in H_G^1(\text{pt}; \mathbb{Z}_2)$, then ${}^\epsilon K_G^*(\text{pt}) = {}^\epsilon R_G \oplus {}^\epsilon R_G^1$, where the graded representation rings were defined last subsection. On the other hand, twisting by $H_G^3(\text{pt}; \mathbb{Z})$ introduces spinors — e.g. $H_{\text{SO}(n)}^3(\text{pt}; \mathbb{Z}) \simeq \mathbb{Z}_2$ and ${}^-K_{\text{SO}(n)}^*(\text{pt}) \simeq R_{\text{SO}(n)}^-$.

We freely interchange K -theory and K -homology, using whichever is more convenient or appropriate, through *Poincaré duality*, which for a compact manifold X says

$${}^\tau K_i^G(X) \simeq {}^{\tau'} K_G^{i+\dim X}(X), \quad (2.4)$$

where $\tau + \tau' = (\text{sw}_1^G(X), \text{sw}_3^G(X)) \in H_G^1(X; \mathbb{Z}_2) \times H_G^3(X; \mathbb{Z})$, involving G -twisted Stiefel–Whitney classes.

For X a compact manifold fixed pointwise by G , and H a subgroup of G , we know ${}^\tau K_G^*(X \times G/H) = {}^\tau K_H^*(X)$ (see [61]) and hence

$${}^\tau K_*^G(X \times G/H) = {}^{\tau'} K_{*+\dim G+\dim H}^H(X) \quad (2.5)$$

by Poincaré duality, for the appropriate twist τ' . If N is a normal subgroup of G , and N acts freely on X , then using (2.2)

$${}^\tau K_*^G(X) = {}^\tau K_*^{G/N}(X/N). \quad (2.6)$$

If instead X is fixed by H and the twist $\tau \in H_{G \times H}^1(X; \mathbb{Z}_2) \times H_{G \times H}^3(X; \mathbb{Z})$ lies in $(H_G^1(X; \mathbb{Z}_2) \otimes 1) \times (H_G^3(X; \mathbb{Z}) \otimes 1)$ (recall (2.1)), then

$${}^\tau K_{G \times H}^*(X) = R_H \otimes_{\mathbb{Z}} {}^\tau K_G^*(X). \quad (2.7)$$

Let U be a G -invariant open subset of X , and τ' and τ'' be the restrictions of the twist τ on X to U and X/U , respectively. Then the *six-term exact sequence* says:

$$\begin{array}{ccccc} {}^{\tau'} K_0^G(U) & \longleftarrow & {}^\tau K_0^G(X) & \longleftarrow & {}^{\tau''} K_0^G(X/U) \\ \downarrow & & & & \uparrow \\ {}^{\tau''} K_1^G(X/U) & \longrightarrow & {}^\tau K_1^G(X) & \longrightarrow & {}^{\tau'} K_1^G(U) \end{array} \quad (2.8)$$

The Hodgkin spectral sequence (see e.g. [22, Sect 3.3]) explains how to restrict G -equivariance in K -theory to a subgroup H . It starts from a closed subgroup H of a Lie group G with torsion-free π_1 , and a G -action on space X , and defines $E_2^{p,q} = \text{Tor}_{R_G}^p(R_H, {}^\tau K_G^q(X))$; the resulting spectral sequence of R_G -modules strongly converges to ${}^{\tau'} K_H^*(X)$, where τ' is the image of τ under the natural map $H_G^3(X; \mathbb{Z}) \rightarrow H_H^3(X; \mathbb{Z})$ restricting G -equivariance to H .

It is useful to have explicit descriptions of the Dixmier–Douady bundles representing the twists relevant to these K -groups ${}^\tau K_G^*(X)$. These are bundles over X with fibre the compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H})$ on a G -stable Hilbert space \mathcal{H} , where we usually require $\mathcal{H} \simeq \mathcal{H} \otimes L^2(G)$ (i.e. each G -irrep appears in \mathcal{H} with infinite multiplicity). Many of these were constructed in [22, Sect.2.2]; we'll quickly sketch two of them. Consider first ${}^k K_{\mathbb{T}^{\text{ad}}}^*(\mathbb{T})$ for the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: here we can simply take $\mathcal{H} = L^2(\mathbb{T})$. Let $U_k \in U(\mathcal{H})$ be the unitary operator defining the equivalence $\pi \otimes a^k \simeq \pi$, where π is the regular representation of \mathbb{T} and $a^k \in R_{\mathbb{T}}$. The sections of our bundle are maps $f : \mathbb{R} \rightarrow \mathcal{K}$, continuous on $[0, 1]$, satisfying $f(t) = U_k f(t+1) U_k^*$ for all $t \in \mathbb{R}$. The \mathbb{T} -action is defined by $(t.f)(s) = \text{Ad}(\pi(t))f(s)$.

Consider next ${}^k K_{G^{\text{ad}}}^*(G)$ for $G = \text{SU}(2)$. Here, $\mathcal{H} = L^2(G) \otimes \ell^2$, with regular G -representation $\pi \simeq \sum_n \sigma_n \otimes 1_\infty$. The conjugacy classes of G are parametrised by the points on half of a maximal torus T (say $\text{diag}(e^{it}, e^{-it})$ for $0 \leq t \leq 1$). Cover G with G -invariant open sets D_1, D_2 containing $I, -I$ respectively. On each fibre $(x, c)_i$, $x \in D_i$, $c \in \mathcal{K}$, we define the G -action $g.(x, c)_i = (gxg^{-1}, \text{Ad}(\pi_g)c)_i$. Define $U_k \in U(\mathcal{H})$ as in [22]; restricting π to the maximal torus T , i.e. considering weight spaces $V_{m, n-1}$, $m = n, n-2, \dots, 2-n, -n$ (for convenience write $V_{m, -n+1}$ when $m < 0$), U_k acts as the shift operator $V_{m, n-1} \rightarrow V_{m-k, n-1-k}$. For $x \in D_1 \cap D_2 \cap S$, the gluing condition is $(gxg^{-1}, c)_1 = (gxg^{-1}, \text{Ad}(\pi_g U_k \pi_g^{-1})c)_2$. Then for $x \in D_1 \cap D_2 \cap S$, when $g \in Z_G(x) = T$, $\pi_g U_k \pi_g^{-1} = \lambda_k U_k$ where λ_k is the character $a^k \in R_T$ (consistency requires that it be a character).

Some of our *nimreps* involve the product ${}^\tau K_0^{G^{\text{ad}}}(G) \times {}^{\tau'} K_0^{G^{\text{ad}}}(G/Z) \rightarrow {}^{\tau'} K_0^{G^{\text{ad}}}(G/Z)$, for Z a subgroup of the centre of G (the relation between the twists τ, τ' is described in section 5.2), which arises through the push-forward of the products $g * g' \mapsto gg'$ and $g * h \mapsto \phi(g)h$ on the spaces (where ϕ is the projection $G \rightarrow G/Z$) — these are clearly G -equivariant. It will sometimes be convenient to rewrite ${}^{\tau'} K_0^{G^{\text{ad}}}(G/Z)$ as ${}^{\tau'} K_0^{G^{\text{ad}} \times Z^L}(G)$, but the product in this case is again given by $g * g' \mapsto gg'$. Some of our *full systems* involve a product on ${}^\tau K_0^{G^{\text{ad}} \times Z^L \times Z^R}(G)$ or ${}^\tau K_0^{G^{\text{ad}} \times Z^L}(G)$, making them into rings; this arises through the convolution product $g * g' \mapsto \bigoplus_{z \in Z} gzg'$ (at least when Z is finite). The product ${}^\tau K_{G_1^L \times G_2^R}^0(H \times H) \times {}^{\tau'} K_{G_2^L \times G_3^R}^0(H \times H) \rightarrow {}^{\tau''} K_{G_1^L \times G_3^R}^0(H \times H)$, for subgroups G_i of a *finite* group $H \times H$, arises through the convolution product $(h_1, h_2) * (h'_1, h'_2) \mapsto \bigoplus_{g \in G_2} (h_1, h_2)g(h'_1, h'_2)$ — the relation of the twists τ, τ', τ'' is explained in [21]. For Lie groups this product is more subtle, and should be meant in the sense of the Jeffrey–Weitsman product [47] on conjugacy classes.

In twisted K -theory, the naive product would add the twists. A twist-preserving product arises for instance when the twist is transgressed [63], i.e. lies in the image of the appropriate map $H_G^i(\text{pt}; A) \rightarrow H_G^{i-1}(X; A)$. A twist $\tau \in H_G^3(G; \mathbb{Z})$ only determines the K -theory ${}^\tau K_G^*(G)$ as an additive group; the ring structure comes from a choice of lift (if it exists) of the 3-cocycle τ to $H_G^4(\text{pt}; \mathbb{Z})$. But for G compact, connected, simply-connected, transgression identifies $H_G^4(\text{pt}; \mathbb{Z})$ with $H_G^3(G; \mathbb{Z})$ so τ parametrizes the full ring structure.

We can see directly the action of the k th roots of unity in \mathbb{T} on the K -group ${}^k K_{\mathbb{T}}^1(\mathbb{T})$. For this purpose, instead of placing a copy of the twisting unitary $U_k = U_1^k$ at each integer, it is more convenient to place U_1 at each point in $\frac{1}{k}\mathbb{Z}$, as was done at the end of [22, Sect.3.1]. A section now consists of k continuous maps $f_l : [0, \frac{1}{k}] \rightarrow \mathcal{K}$ for $l \in \mathbb{Z}_k$, related by $f_l(\frac{1}{k}) = \text{Ad}(U_1)f_{l+1}(0)$ for all l . Recall from [22, Sect.3.1] the identification of the basis of ${}^k K_{\mathbb{T}}^1(\mathbb{T})$ as roots of unity in \mathbb{T} . We get a \mathbb{Z}_k -action on this bundle, and hence on ${}^k K_{\mathbb{T}}^1(\mathbb{T})$, by sending section (f_0, \dots, f_{k-1}) to $(f_l, f_{l+1}, \dots, f_{l+k-1})$, or equivalently fibre (t, c) to $(t + \frac{l}{k}, c)$.

2.3 Rational conformal field theory and subfactors

For a review with references of the basic algebraic structure of (unitary) rational conformal field theories (RCFT), see e.g. [41]. The chiral data of an RCFT consists of a rational vertex operator algebra (VOA) \mathcal{V} whose category of modules forms a modular tensor category. The finitely many irreducible \mathcal{V} -modules (the simple objects in the category) are called the *primaries* $\lambda \in \Phi$. The (commutative associative semi-simple) Grothendieck ring of this category, with preferred basis Φ , is called the *fusion ring* or *Verlinde ring* $\text{Ver}(\mathcal{V})$, where we write $\lambda\mu = \sum_{\nu \in \Phi} N_{\lambda,\mu}^\nu \nu$. The primary corresponding to \mathcal{V} -module \mathcal{V} itself, is the unit in $\text{Ver}(\mathcal{V})$, denoted 1 and called the *vacuum*. The *simple-currents* $j \in \Phi$ are the invertible primaries, i.e. those for which there is a $j' \in \Phi$ such that $jj' = 1$. Multiplication in $\text{Ver}(\mathcal{V})$ by a simple-current permutes the primaries. The simple-currents form a group, by composition of those permutations; if j, j' are two simple-currents, then

$$N_{j\lambda, j'\mu}^{jj'\nu} = N_{\lambda,\mu}^\nu. \quad (2.9)$$

The modular tensor category comes with unitary representations of the braid groups and mapping class groups. In particular we get a unitary representation of the modular group $\text{SL}(2, \mathbb{Z})$ on the complexification $\mathbb{C} \otimes_{\mathbb{Z}} \text{Ver}(\mathcal{V})$: write $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T$, a diagonal matrix. *Charge-conjugation* $C = S^2$ is an involution on $\text{Ver}(\mathcal{V})$, which permutes the primaries; then $S_{C\lambda, \mu} = S_{\lambda, C\mu} = S_{\lambda, \mu}^*$ and $T_{C\lambda, C\mu} = T_{\lambda, \mu}$ for all primaries λ, μ . For any simple-current j , there is a function $Q_j : \Phi \rightarrow \mathbb{Q}$ such that $S_{j\lambda, \mu} = \exp[2\pi i Q_j(\mu)] S_{\lambda, \mu}$. The simple-currents are precisely those primaries $j \in \Phi$ whose *quantum-dimension* $S_{j,1}/S_{1,1}$ equals 1. The *Verlinde formula* says that the matrices $N_\lambda = (N_{\lambda, \mu}^\nu)$ are simultaneously diagonalised by S and have eigenvalues $S_{\lambda, \mu}/S_{1, \mu}$, of multiplicity 1 for each $\mu \in \Phi$. The Perron-Frobenius eigenvalue of the nonnegative matrix N_λ is the quantum-dimension $S_{\lambda,1}/S_{1,1}$.

Definition 1. A matrix $\mathcal{Z} = (\mathcal{Z}_{\lambda, \mu})_{\lambda, \mu \in \Phi}$ is called a modular invariant if $\mathcal{Z}S = S\mathcal{Z}$, $\mathcal{Z}T = T\mathcal{Z}$, each entry $\mathcal{Z}_{\lambda, \mu}$ is a nonnegative integer, and $\mathcal{Z}_{1,1} = 1$.

The modular invariant describes how the space of states of the RCFT carries a representation of the VOA. For example, $\mathcal{Z} = I$ and $\mathcal{Z} = C$ are both modular invariants; when \mathcal{Z} is a permutation matrix it is called an *automorphism invariant*. Automorphism invariants \mathcal{Z} are special among modular invariants in that matrix multiplication $\mathcal{Z}\mathcal{Z}'$ or $\mathcal{Z}'\mathcal{Z}$ are modular invariants iff \mathcal{Z}' is.

If j is an order- n simple-current (i.e. the permutation of primaries corresponding to j is order n) then $T_{j,j}/T_{1,1}$ will be a $2n$ -th root of unity; when $T_{j,j}/T_{1,1}$ is also order- n , then a modular invariant $\mathcal{Z}_{\langle j \rangle}$ can be associated to $\langle j \rangle \simeq \mathbb{Z}_n$; these are called the *simple-current invariants* (see e.g. [60]). In this case write $T_{j,j}/T_{1,1} = \exp[2\pi i h_j]$ for $h_j \in \frac{1}{n}\mathbb{Z}$. Then

$$(\mathcal{Z}_{\langle j \rangle})_{\lambda, \mu} = \sum_{1 \leq l \leq n} \delta^{\mathbb{Z}}(Q_j(\lambda) - l h_j) \delta_{\mu, j^l \lambda}, \quad (2.10)$$

where $\delta^{\mathbb{Z}}(x)$ equals 1 or 0 depending on whether or not x is integral. $\mathcal{Z}_{\langle j \rangle}$ will be an automorphism invariant iff the root of unity $T_{j,j}/T_{1,1}$ has order exactly equal to that of the permutation j .

Another source of modular invariants are the *conformal embeddings*, which are VOAs \mathcal{V}' containing \mathcal{V} but with the same central charge. The corresponding modular invariant is built from the branching rules expressing irreducible \mathcal{V}' -modules as direct sums of \mathcal{V} -modules. The result is a block-diagonal modular invariant.

This data (the basis Φ , the ring $\text{Ver}(\mathcal{V})$, and the matrices S, T, \mathcal{Z}) helps define the *bulk CFT*, which describes *closed* string theory. *Boundary CFT* (see e.g. the review [59]), describing *open* strings, starts with a finite set of *boundary states* $x \in \mathcal{B}$. The Verlinde ring acts on the \mathbb{Z} -span of these x , and this module structure is called a *nimrep*, written $\lambda x = \sum_{y \in \mathcal{B}} \mathcal{N}_{\lambda,x}^y y$. More precisely:

Definition 2. A set of matrices $\mathcal{N}_{\lambda} = (\mathcal{N}_{\lambda,x}^y)_{x,y \in \mathcal{B}}$, for all $\lambda \in \Phi$, is called a nimrep if all entries $\mathcal{N}_{\lambda,x}^y$ are nonnegative integers, $\mathcal{N}_{\lambda} \mathcal{N}_{\mu} = \sum_{\nu \in \Phi} N_{\lambda,\mu}^{\nu} \mathcal{N}_{\nu}$, $\mathcal{N}_{C\lambda} = \mathcal{N}_{\lambda}^t$, and given any $x, y \in \mathcal{B}$ there is a $\lambda \in \Phi$ such that $\mathcal{N}_{\lambda,x}^y \neq 0$. Two nimreps $\mathcal{N}, \mathcal{N}'$ are called equivalent if there is a permutation matrix P such that $P \mathcal{N}_{\lambda} P^{-1} = \mathcal{N}'_{\lambda}$.

P defines the identification of the boundary states \mathcal{B} and \mathcal{B}' . An analogue of the Verlinde formula holds, namely

$$\mathcal{N}_{\lambda,x}^y = \sum_{(\mu,i)} \Psi_{x,(\mu,i)} \frac{S_{\lambda,\mu}}{S_{1,\mu}} \Psi_{y,(\mu,i)}^*, \quad (2.11)$$

saying that the matrices $\mathcal{N}_{\lambda} = (\mathcal{N}_{\lambda,x}^y)$ can be simultaneously diagonalised by some unitary matrix Ψ , with eigenvalues $S_{\lambda,\mu}/S_{1,\mu}$, where now each $\mu \in \Phi$ comes with some multiplicity m_{μ} (possibly 0) — this multi-set is called the *exponent* of nimrep \mathcal{N} . In particular, the vacuum has multiplicity 1; the nonnegative matrix \mathcal{N}_{λ} will have Perron-Frobenius eigenvalue equal to the quantum-dimension $S_{\lambda,1}/S_{1,1}$. For example, the matrices \mathcal{N}_j associated to any simple-current j will be a permutation matrix (since the only diagonalisable nonnegative integer matrices with largest eigenvalue 1 are permutation matrices).

The nimrep will be compatible with the modular invariant \mathcal{Z} , in the sense the eigenvalue multiplicity m_{μ} equals $\mathcal{Z}_{\mu,\mu}$ for all $\mu \in \Phi$. For example the Verlinde ring is itself a nimrep, called the *Verlinde nimrep*, with $\mathcal{N}_{\lambda} = N_{\lambda}$ and $\mathcal{B} = \Phi$, and is compatible in this sense with the diagonal modular invariant $\mathcal{Z} = I$.

An important example is the Drinfeld double of finite groups G . Throughout this paper we will restrict to the simpler case of trivial 3-cocycle twist, as finite groups are not our primary interest. The primaries are pairs (g, χ) where g is a representative of a conjugacy class in G and χ is an irreducible representation (*irrep*) of the centraliser $Z_g(G)$. The vacuum 1 is $(1, 1)$. We'll write $\text{Ver}(G)$ for its Verlinde ring.

The other important example for us is the loop group LG at level $k \in \mathbb{Z}_{>0}$, where G is a compact, connected, simply-connected Lie group. We write $\text{Ver}_k(G)$ for its Verlinde ring and $P_+^k(G)$ for its primaries Φ . In particular, for G of rank r , $\lambda \in P_+^k(G)$ is the affine highest-weight $\lambda = (\lambda_0; \lambda_1, \dots, \lambda_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ where $\sum a_i^{\vee} \lambda_i = k$ for positive

integers a_i^\vee (the *co-labels*) depending only on G . For all G , $a_0^\vee = 1$ and $(k; 0, \dots, 0)$ denotes the vacuum 1. $\text{Ver}_k(G)$ can be expressed as $R_G/I_k(G)$ for some ideal $I_k(G)$ of R_G called the *fusion ideal*, and primary λ is associated to class $[\rho_{\bar{\lambda}}] \in R_G/I_k(G)$ where $\rho_{\bar{\lambda}}$ is the G -irrep with highest-weight $\bar{\lambda} = (\lambda_1, \dots, \lambda_r)$. For example, for $G = \text{SU}(n)$, we have all $a_i^\vee = 1$, there are n simple-currents, and charge-conjugation is nontrivial iff $n > 2$.

Restrict now to the loop group setting, for G a Lie group of rank $r > 0$, and fix a modular invariant \mathcal{Z} and compatible nimrep \mathcal{N}_λ . Equivalence classes of D -branes (extended structures in space-time on which the end-points of open strings reside) are parametrised by the boundary states $x \in \mathcal{B}$. The dynamics of the branes is controlled by their *conserved charges*, i.e. an assignment of an integer q_x to each $x \in \mathcal{B}$ and a choice of integer M such that

$$\dim(\bar{\lambda}) q_x \equiv \sum_{y \in \mathcal{B}} \mathcal{N}_{\lambda, x}^y q_y \pmod{M} \quad (2.12)$$

is satisfied for each primary $\lambda = (\lambda_0; \lambda_1, \dots, \lambda_r)$ and boundary state x , where $\dim(\bar{\lambda})$ denotes the dimension of the G -irrep with highest-weight $(\lambda_1, \dots, \lambda_r)$. The rescaled assignment $(\{nq_x\}, nM)$, for any nonzero integer n , is regarded as equivalent to $(\{q_x\}, M)$; the set of all equivalence classes of assignments $(\{q_x\}, M)$ satisfying (2.12) forms a \mathbb{Z} -module $\mathcal{M}_\mathcal{N}$ called the *charge-group*, which can be regarded as the universal solution to (2.12) for the nimrep \mathcal{N} . The unfortunate restriction here to CFTs associated to loop groups is because the analogue of $\dim(\bar{\lambda})$ is not clear for other RCFT (perhaps it involves the *special subspace* of Nahm [55], as that should give an upper bound on the quantum-dimensions $S_{\lambda,1}/S_{1,1}$ — we thank Matthias Gaberdiel for communication on this point). In section 7 we make a definite proposal for G a finite group.

For example, consider the Verlinde nimrep $\mathcal{N}_\lambda = N_\lambda$. If we start with $q_1 = 1$, then (2.12) forces $q_\lambda = \dim(\bar{\lambda})$ for each primary λ , and the largest possible M which works is $\widetilde{M} = \gcd_{\bar{\mu} \in I_k(G)} \dim(\bar{\mu})$, where $I_k(G)$ is the fusion ideal. More generally, any solution to (2.12) will satisfy $q_\lambda = q_1 \dim(\bar{\lambda})$; this implies we can rescale it so that $M = \widetilde{M}$, and that charge-assignment is then recovered (up to equivalence) by the value $q_1 \in \mathbb{Z}_{\widetilde{M}}$. Hence the charge-group $\mathcal{M}_\mathcal{N}$ is simply $\mathbb{Z}_{\widetilde{M}}$. A consequence of this discussion is that the charge-group $\mathcal{M}_\mathcal{N}$ for any nimrep \mathcal{N} of $\text{Ver}_k(G)$ must be \widetilde{M} -torsion.

Subfactor theory captures and enhances this data from bulk and boundary CFT. We refer to [48, 24] for the basic theory of subfactors etc, and [9, 7, 8] for the theory of alpha-induction etc.

Given a type III factor N , let ${}_N\mathcal{X}_N$ denote a finite system of endomorphisms on N [9, Defn 2.1]. Write $\Sigma({}_N\mathcal{X}_N)$ for the endomorphisms which decompose into a finite number of irreducibles from ${}_N\mathcal{X}_N$. For $\lambda, \rho \in \Sigma({}_N\mathcal{X}_N)$, write $\langle \lambda, \rho \rangle$ for the dimension of the intertwiner space $\text{Hom}(\lambda, \rho)$. The *sector* $[\lambda]$ identifies all endomorphisms $\text{Ad}(u)\lambda$ for unitaries u in the target algebra.

Suppose ${}_N\mathcal{X}_N$ is nondegenerately braided [9, Sect.2.2]. Among other things this means $\lambda, \mu \in {}_N\mathcal{X}_N$ commute up to a unitary $\epsilon = \epsilon(\lambda, \mu)$, i.e. $\lambda\mu = \text{Ad}(\epsilon)\mu\lambda$, and these unitaries $\{\epsilon(\lambda, \mu)\}$ can be chosen to satisfy the braiding–fusion relations. The $\text{SL}(2, \mathbb{Z})$ -representation, i.e. its generators S, T , is obtained here from the intertwiners associated to the Hopf link and twist in the manner familiar from modular tensor categories [64]; the primaries are the sectors in ${}_N\mathcal{X}_N$ with vacuum 1 being the identity endomorphism, and the Verlinde ring is given by composition: $[\lambda][\mu] = \sum N_{[\lambda], [\mu]}^{[\nu]}[\nu]$ where $N_{[\lambda], [\mu]}^{[\nu]} = \langle [\lambda][\mu], [\nu] \rangle$.

The data of full and boundary CFT requires a subfactor $N \subset M$. Let $\iota : N \rightarrow M$ be the inclusion and $\bar{\iota} : M \rightarrow N$ its conjugate. Then $\theta = \bar{\iota}\iota$ is called the *canonical endomorphism* and $\gamma = \bar{\iota}\iota$ the *dual canonical endomorphism*. We require θ to be in $\Sigma({}_N\mathcal{X}_N)$. Using the braiding $\epsilon^+ := \epsilon$ or its opposite $\epsilon^- := \epsilon^{-1}$, we can lift an endomorphism $\lambda \in {}_N\mathcal{X}_N$ of N to one of M : $\alpha_\lambda^\pm := \gamma^{-1}\text{Ad}(\epsilon^\pm(\lambda, \theta))\lambda\gamma$. Then $\mathcal{Z}_{\lambda, \mu} := \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ is a modular invariant [9, 20]. The induced $\alpha^\pm({}_N\mathcal{X}_N)$ generate the *full system* ${}_M\mathcal{X}_M$. These maps α^\pm from the Verlinde ring ${}_N\mathcal{X}_N$ to the full system ${}_M\mathcal{X}_M$ are called *alpha-inductions*; they preserve multiplication, addition, and (charge-)conjugation. By ${}_N\mathcal{X}_M$ (resp. ${}_M\mathcal{X}_N$) we mean all irreducible sectors appearing in any $\iota\lambda$ (resp. $\lambda\bar{\iota}$) for $\lambda \in {}_N\mathcal{X}_N$. The nimrep is the ${}_N\mathcal{X}_N$ action on ${}_N\mathcal{X}_M$, given by composition; it will automatically be compatible with the given modular invariant [9, 10]. This nimrep product is one of 8 (6 independent) natural products (compositions) ${}_P\mathcal{X}_Q \times {}_Q\mathcal{X}_R \rightarrow {}_P\mathcal{X}_R$ among the sectors, one for each triple $P, Q, R \in \{M, N\}$.

Unlike ${}_N\mathcal{X}_N$, the full system ${}_M\mathcal{X}_M$ is not in general nondegenerately braided. However its subsystem ${}_M\mathcal{X}_M^0$, defined to be the intersection of all irreducible sectors in $\alpha^+({}_N\mathcal{X}_N)$ with all those in $\alpha^-({}_N\mathcal{X}_N)$, is. ${}_M\mathcal{X}_M^0$ is called the *neutral system*, and just as ${}_N\mathcal{X}_N$ captures the chiral data of some VOA \mathcal{V} , ${}_M\mathcal{X}_M^0$ captures that of the maximally extended VOA $\mathcal{V}' \subseteq \mathcal{V}$ compatible with the modular invariant \mathcal{Z} . *Sigma-restriction* $\sigma_\beta := \bar{\iota} \circ \beta \circ \iota$ maps the full system ${}_M\mathcal{X}_M$ to $\Sigma({}_N\mathcal{X}_N)$; it preserves addition and conjugates but not multiplication (after all, σ takes the identity in ${}_M\mathcal{X}_M$ to the canonical endomorphism θ , not the identity in ${}_N\mathcal{X}_N$). Applied to sectors in the neutral system ${}_M\mathcal{X}_M^0$, it coincides in the VOA language with the restriction (*branching rules*) of \mathcal{V}' -modules to \mathcal{V} .

When a modular invariant \mathcal{Z} has at least one compatible full system, nimrep etc, we call it *sufferable* [5, 6, 7, 8, 9, 10, 20, 25, 26, 27, 56, 57, 70]. In the finite group setting [58, 21], the sufferable modular invariants are parametrized by pairs (H, ψ) where H is a subgroup of $G \times G$, and $\psi \in H_H^2(\text{pt}; \mathbb{T})$ here plays the role of discrete torsion: then $\beta_g(h) := \psi(g, h)\psi(h, g)^{-1}$ is a 1-dimensional representation of $Z_H(g)$ and can be used to twist a modular invariant, for example. The diagonal modular invariant corresponds to the choice $H = \Delta = \{(g, g) : g \in G\}$ and $\psi = 1$. Sections 3.2 and 5.1 include examples of $\psi \neq 1$. No such parametrisation of sufferable modular invariants is known in the loop group setting, though no $\text{SU}(n)$ modular invariant is known to us to be insufferable (there are plenty of insufferable modular invariants for other Lie groups G however).

Consider for concreteness $G = \text{SU}(2)$ at level k , with $k+1$ primaries $\lambda = (\lambda_0; \lambda_1)$

for $\lambda_0 + \lambda_1 = k$ and $0 \leq \lambda_i \leq k$. Its Verlinde ring $\text{Ver}_k(G)$ is the quotient of R_G by the fusion ideal (σ_{k+2}) , where the class $[\sigma_i]$ corresponds to primary $(k+1-i; i-1)$. The vacuum is $(k; 0)$. Charge-conjugation here is trivial and there is one nontrivial simple-current, $j = (0; k)$, corresponding to permutation $j(\lambda_0; \lambda_1) = (\lambda_1; \lambda_0)$ with $Q_j(\lambda) = \lambda_1/2$ and $h_j = k/4$. Its modular invariants are classified in [14] and all are sufferable; they fall into an A-D-E pattern:

- (i) For any k , \mathbb{A}_{k+1} corresponds to the diagonal modular invariant $\mathcal{Z} = I$. The nimrep and full system are the Verlinde ring $\text{Ver}_k(G) = {}^{k+2}K_G^1(G)$ and both alpha-inductions are the identity. The \mathbb{A}_{k+1} diagram describes the multiplication in $\text{Ver}_k(G)$ by the fundamental weight $\sigma = \sigma_2 \in R_G$ (which generates $\text{Ver}_k(G) \simeq \mathbb{Z}[\sigma]/(\sigma_{k+2})$) on the preferred basis $[\sigma_1], [\sigma_2], \dots, [\sigma_{k+1}]$. The D-brane charge-group $\mathcal{M}_{\mathbb{A}}$ is \mathbb{Z}_{k+2} , generated by the assignment $[\sigma_l] \mapsto q_{[\sigma_l]} = \dim(\sigma_l) = l$ for $\sigma_l \in R_G$ (modulo $k+2$ this dimension is well-defined).
- (ii) For any even k , the Dynkin diagram $\mathbb{D}_{k/2+2}$ likewise describes the nimrep corresponding to the simple-current invariant $\mathcal{Z}_{\langle j \rangle}$, again with respect to the preferred basis. When $k/2$ is odd (resp. even), the modular invariant is an automorphism invariant (resp. in block-diagonal form), the full system is $\text{Ver}_k(G)$ [10] (resp. two copies of $\mathbb{D}_{k/2+2}$ [5, III]), and the charge-group $\mathcal{M}_{\mathbb{D}}$ is \mathbb{Z}_4 (resp. $\mathbb{Z}_2 \times \mathbb{Z}_2$).
- (iii) The diagram \mathbb{E}_6 defines the nimrep corresponding to the conformal embedding $\text{SU}(2)_{10} \rightarrow \text{Sp}(4)_1$. Its full system etc is given in [5, III]. The charge-group is \mathbb{Z}_3 .
- (iv) The diagram \mathbb{E}_7 defines the nimrep corresponding to a \mathbb{Z}_2 -orbifold of the \mathbb{D}_{10} simple-current invariant at $k = 16$. We discuss this example in section 6.2 below. The charge-group is \mathbb{Z}_2 .
- (v) The diagram \mathbb{E}_8 defines the nimrep corresponding to the conformal embedding $\text{SU}(2)_{28} \rightarrow G_{2,1}$. Its full system etc is given in [5, III]. The charge-group is trivial.

2.4 K -theory and CFT

Consider first G a finite group. This case has been an indispensable guide to the generalisations of Freed-Hopkins-Teleman. We restrict attention here to trivial H^3 -twist — see [21] for a more complete treatment. The Verlinde ring is $K_{G^{\text{ad}}}^0(G) = K_{\Delta_G^L \times \Delta_G^R}^0(G \times G)$. This recovers naturally the preferred basis described in section 2.3.

Fix now a sufferable modular invariant, i.e. a subgroup H of $G \times G$ and some $\psi \in H_H^2(\text{pt}; \mathbb{T})$. Assume for now that ψ is trivial. The K -theory of the full system is $K_{H^L \times H^R}^0(G \times G)$. The boundary states $x \in \mathcal{B}$ are the natural basis of $K_{H^L \times \Delta_G^R}^0(G \times G) = K_H^0(G)$ where $(h_1, h_2).g = h_1 g h_2^{-1}$; the $x \in \mathcal{B}$ therefore consist of a pair $(H(g_1, g_2)\Delta_G, \phi)$ of an $H \times \Delta_G$ -orbit in $G \times G$, and an irrep ϕ of the stabiliser in $H \times \Delta_G$ of (g_1, g_2) . The special element in the nimrep, corresponding in the subfactor language

to the inclusion $\iota : N \subset M$ of factors, is $\iota = (H\Delta_G, 1)$. Likewise, the conjugate map, the surjection $\bar{\iota} : M \rightarrow N$, corresponds to $\bar{\iota} = (\Delta_G H, 1) \in K_{\Delta_G^L \times H^R}^0(G \times G)$. When the 2-cocycle ψ is not trivial, the $R_{H \times H}$ and $R_{H \times G}$ module structures of the full system $K_{H^L \times H^R}^0(G \times G)$ and nimrep $K_{H^L \times \Delta_G^R}^0(G \times G)$ resp. are unchanged, but the bundles, alpha-induction, sigma-restriction, the product in ${}_M\mathcal{X}_M$, the $\text{Ver}(G)$ -module structure of ${}_M\mathcal{X}_N$, etc will be twisted by ψ . We'll see examples of nontrivial ψ later in the paper.

We are more interested in this paper in the case of Lie groups (of dimension > 0), where considerably less is known. The main result we build on is the expression of the Verlinde ring as a K -group. Let G be a compact group (not necessarily connected or simply-connected), with identity connected component G_1 , and fix some element $f \in G$. Write $[fG_1]$ for the conjugacy class in G containing fG_1 , and $L_f G$ for the f -twisted loop group consisting of all maps $\gamma : \mathbb{R} \rightarrow G$ satisfying $\gamma(t+1) = f\gamma(t)f^{-1}$. Fix any twist $\tau \in H_G^1([fG_1]; \mathbb{Z}_2) \times H_G^3([fG_1]; \mathbb{Z})$, and let ${}^\tau R(L_f G)$ denote the space of admissible representations — see [33] for its definition and its interpretation using highest-weight modules for the corresponding f -twisted affine algebra. Then the main theorem of Freed-Hopkins-Teleman is:

Theorem 1. [33] *There is a natural isomorphism*

$$\tau + \sigma K_{G^{\text{ad}}}^d([fG_1]) \simeq {}^\tau R(L_f G), \quad (2.13)$$

where d is the dimension of the centraliser $Z_G(f)$, and the twist σ is the cocycle for the projective action of $L_f G$ on the graded Clifford algebra $\text{Cliff}(\mathfrak{g}^*)$. Moreover, $\tau + \sigma K_{G^{\text{ad}}}^{d+1}([fG_1]) \simeq {}^\tau R^1(L_f G)$, the corresponding space of graded representations.

The relation between K -theory and branes is reviewed in [53]. Consider a compact, connected, simply-connected Lie group G of rank r , at level k , and write $\kappa = k + h^\vee$ where h^\vee is the dual Coxeter number. This corresponds to the diagonal modular invariant and Verlinde nimrep. The charge-group $\mathbb{Z}_{\widetilde{M}}$ (recall section 2.3) is related to the twisted *nonequivariant* K -homology ${}^\kappa K_*(G)$ (see Conjecture 1(a) below). The spectral sequences argument of section 2.1 identifies $H_{G^{\text{ad}}}^3(G; \mathbb{Z}) \simeq \mathbb{Z}$ with $H^3(G; \mathbb{Z})$. These groups ${}^\kappa K_*(G)$ can be computed (via Poincaré duality) from ${}^\kappa K_{G^{\text{ad}}}^*(G)$ using the Hodgkin spectral sequence (recall section 2.2) with $H = 1$, together with the resolution of $\text{Ver}_k(G)$ found in e.g. [52]; the result [13, 17] can be expressed as ${}^\kappa K_0(G) = \oplus_{l \text{ even}} \text{Tor}_l$, ${}^\kappa K_1(G) = \oplus_{l \text{ odd}} \text{Tor}_l$, where Tor_l is identified with the degree- l part of the exterior algebra $\bigwedge \{x_1, \dots, x_{r-1}\}$. More explicitly, ${}^\kappa K_i(G) \simeq \mathbb{Z}_M^{2^{r-2}}$ for all i (except for $G = \text{SU}(2)$, where ${}^\kappa K_i(\text{SU}(2)) \simeq \mathbb{Z}_M, 0$ for i even, odd respectively), for some integer $M = M_k(G)$ depending on G and κ . For example, $M_k(\text{SU}(n)) = \frac{\kappa}{\gcd(\kappa, y)}$ for $y = \text{lcm}(1, 2, \dots, n-1)$.

Conjecture 1(a) *For all simply-connected, compact, connected G and any level k , the integer $M_k(G)$ appearing in the groups ${}^\kappa K_*(G)$ equals the gcd of the dimensions of all weights ν in the fusion ideal $I_k(G)$, or equivalently the charge-group \mathcal{M}_N for the Verlinde nimrep $\mathcal{N}_\lambda = N_\lambda$ is isomorphic to $\mathbb{Z}_{M_k(G)}$.*

(b) The largest number M (call it M_k^{spin}) such that

$$\dim(\lambda) \dim(\mu) \equiv \sum_{\nu \in \Phi} N_{\lambda, \mu}^{\nu} \dim(\nu) \pmod{M} \quad (2.14)$$

holds for all $\mathrm{SO}(2r+1)$ non-spinor dominant weights λ and all level k $\mathrm{SO}(2r+1)$ -spinors μ (i.e. μ_r is odd and λ_r is even), is 2^r times the integer $M_k(\mathrm{Spin}(2r+1))$ appearing in Conjecture 1(a).

It is clear the K -theory $M_k(G)$ divides the gcd in Conjecture 1(a). The cases $G = \mathrm{SU}(n)$ and $G = \mathrm{Sp}(2n)$ of Conjecture 1(a) have been proved in [11]; relatively simple expressions for the K -theoretic M are found in [18], and extremely simple formulas for M are conjectured in e.g. [11]. Conjecture 1(b) is needed in section 4 below; it was proved in [38] whenever 4 doesn't divide $M_k(\mathrm{SO}(2r+1))$, where it was also shown $M_k^{spin} = 2^i M_k(\mathrm{Spin}(2r+1))$ for some integer $0 \leq i \leq r$. There is considerable numerical evidence supporting Conjecture 1.

The assignment $x \mapsto q_x$ of charges here is given by the map ${}^{\kappa}K_*^G(G) \rightarrow {}^{\kappa}K_*(G)$ forgetting the G -equivariance. This can be calculated through the commuting diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_k(G) & \longrightarrow & R_G = K_0^G(1) & \xrightarrow{\alpha} & {}^{\tau}K_0^G(G) \longrightarrow 0 \\ & & \beta \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & M\mathbb{Z} & \longrightarrow & \mathbb{Z} = K_0(1) & \xrightarrow{\delta} & {}^{\kappa}K_0(G) \end{array} \quad (2.15)$$

$I_k(G) \subset R_G$ is the fusion ideal, and Theorem 1 gives the top line, where the ring homomorphism α is the push-forward of the embedding of the identity 1 in G . The map $\beta : K_0^G(1) \rightarrow K_0(1)$, forgetting G -equivariance, is of course given by dimension. Conjecture 1(a) says β carries $I_k(G)$ surjectively onto $M\mathbb{Z}$. To identify the target of the map δ , compare the Hodgkin spectral sequence for ${}^{\kappa}K_0(G)$, i.e. $E_{p,0}^{\infty} = E_{p,0}^2 = \mathrm{Tor}_p^{RG}(\mathbb{Z}, {}^{\kappa}K_0^G(G))$ (and all other $E_{p,q}^{\infty} = 0$), with that for $K_0(1)$, i.e. $E_{0,0}^{\infty} = E_{0,0}^2 = \mathrm{Tor}_0^{RG}(\mathbb{Z}, K_0^G(1))$ (and all other $E_{p,q}^{\infty} = 0$): we see that the target of δ should be $\mathrm{Tor}_0^{RG}(\mathbb{Z}, {}^{\kappa}K_0^G(G)) = \mathbb{Z} \otimes_{R_G} \mathrm{Ver}_k(G)$, with $\delta[\rho] = 1 \otimes_{R_G} \rho$.

3 Group-like fusion rings

3.1 Finite groups

As explained in section 2.3, the sufferable modular invariants for the finite group G setting are parametrised by pairs (H, ψ) where H is a subgroup of $G \times G$ and $\psi \in H_H^2(\mathrm{pt}; \mathbb{T})$. Here we explain, using the methods of [21] and expanded in [22, 23], how to arrive at a modular invariant \mathcal{Z} when ψ is trivial (and the twist τ on the double $\mathcal{D}(G)$ is also trivial). This is useful in sections 3.2, 4.1, 5.1. In sections 3.2 and 4.1 we also consider H^2 -twists on the subgroups H .

Let G be a finite group, and H a subgroup of $G \times G$. Let $\sigma : G \times G \rightarrow G \times G$ be the flip $(a, b) \mapsto (b, a)$, and $\pi_{\pm} : G \times G \rightarrow G$ be the coordinate maps $(a, b) \mapsto a$ and

b respectively. Then $K_{\pm} = \pi_{\pm}(H)$ are subgroups of G . Let

$$\begin{aligned} N_+ &= \ker \pi_-|_H = H \cap (K_+ \times 1) = H \cap (G \times 1), \\ N_- &= \ker \pi_+|_H = H \cap (1 \times K_-) = H \cap (1 \times G), \end{aligned} \quad (3.1)$$

so that $N_{\pm} \triangleleft H$ and $N_{\pm} \triangleleft K_{\pm}$. Then $H/N_{\pm} \simeq K_{\mp}$ and

$$K_+/N_+ \simeq H/N_-/N_+ \simeq H/N_+/N_- \simeq K_-/N_- ,$$

because N_+ and N_- pairwise commute in $G \times G$. Let $H^{\pm} = \Delta_{K_{\pm}}(N_{\pm} \times 1)$ which are σ -invariant subgroups of $G \times G$. The subgroups H^+, H^- will have extended systems given by the doubles of K_+/N_+ and K_-/N_- , respectively. Write b^{\pm} for their branching coefficients (sigma-restrictions) from $\mathcal{D}(K_{\pm}/N_{\pm})$ to $\mathcal{D}(G)$. Then the modular invariant for the pair $(H, 0)$ is obtained as

$$\mathcal{Z}_{\lambda\mu} = \sum_{\tau} b_{\tau\lambda}^+ b_{\beta(\tau)\lambda}^-$$

where τ runs over the primary fields of K_+/N_+ , and β is the identification of the chiral primary fields via the above isomorphism $K_+/N_+ \simeq K_-/N_-$.

It therefore suffices to determine sigma-restriction for the flip invariant case $H = \sigma H$ where $K_+ = K_-$ and $N_+ = N_-$. Let $K = K_{\pm}$, $N = N_{\pm}$ so that $H = \Delta_K(N \times 1)$; then $N \triangleleft K$, and K can be an arbitrary subgroup of G .

Consider first the case $\Delta_G \subset H \subset G \times G$ so that $K = G$, and $N \triangleleft G$. Then sigma-restriction $K_{G/N}^0(G/N) \rightarrow K_G^0(G)$, is described as follows [22]. A primary field in the G/N theory is labelled $[gN, \chi]$ for a conjugacy class in G/N of a coset gN , and χ a representation of the centralizer $Z_{gN}(G/N)$. The quotient map $\pi : G \rightarrow G/N$ takes $Z_{gn}(G)$ to $Z_{gN}(G/N)$ for any $n \in N$, and $\chi\pi$ is a representation of $Z_{gn}(G)$. The preimage π^{-1} of the conjugacy class of gN will be a disjoint union of conjugacy classes of G , each with a representative of the form gn for $n \in N$ (since N is normal). Then the sigma-restriction is $\sigma[gN, \chi] = \sum_n [gn, \chi\pi]$, where the sum is over the $n \in N$ giving a disjoint union.

Next consider the case $H \subset \Delta_G \subset G \times G$ so that $N = 1$ and $H = \Delta_K$. Here the extended system is K on K and we need sigma-restriction $K_K^0(K) \rightarrow K_G^0(G)$. Take a primary field $[k, \chi]$ where k describes a conjugacy class in K and χ a representation of the centraliser $Z_k(K)$. Then sigma-restriction is given by [23] : $[k, \chi] \mapsto [k, \text{Ind}_{Z_k(K)}^{Z_k(G)} \chi]$.

The general case $\Delta_K(N \times 1) \subset G \times G$ is a gluing of the special cases $\Delta_K \subset \Delta_K(N \times 1) \subset K \times K$ and $\Delta_K \subset \Delta_G \subset G \times G$. Consequently we have sigma-restriction in stages:

$$\begin{aligned} K_{K/N}^0(K/N) &\rightarrow K_K^0(K) \rightarrow K_G^0(G), \\ [kN, \chi] &\mapsto \sum_n [kn, \text{Ind}_{Z_{kn}(K)}^{Z_{kn}(G)} (\chi\pi)], \end{aligned}$$

where $\pi : K \rightarrow K/N$ is the quotient map, kn represents a conjugacy class in K/N , and χ is a representation of the stabiliser $Z_{kn}(K/N)$. The sum over n is as before.

3.2 Finite abelian groups

This section considers the case where all primaries are *simple-currents* (recall the definition in section 2.3). A primary (g, χ) of a finite group G is a simple-current iff g lies in the centre of G and χ is 1-dimensional. This means all primaries will be simple-currents, iff G is abelian. Assume for concreteness for now that G is the cyclic group \mathbb{Z}_p for p prime.

For these G , $H_{G^{\text{ad}}}^3(G; \mathbb{Z}) = 0$ while $H_G^4(\text{pt}; \mathbb{Z}) \simeq \mathbb{Z}_p$, so the transgression τ is trivial. As an R_G -module, ${}^{\tau(\sigma)}K_G^0(G) \simeq R_G^p$ is independent of σ , but as a ring, ${}^{\tau(\sigma)}K_G^0(G)$ is the group ring $\mathbb{Z}[\mathbb{Z}_d \times \mathbb{Z}_{p^2/d}]$ where $d = \gcd(2p, \sigma)$. Fix trivial twist $\sigma = p$ for concreteness, so we can identify the primaries with pairs $(a, b) \in \mathbb{Z}_p^2$, with the group-like fusion product $(a, b)(a', b') = (a + a', b + b')$.

The modular data is $S_{(a,b),(a',b')} = \frac{1}{p} \exp(2\pi i(ab' + a'b)/p)$ and $T_{(a,b),(a,b)} = \exp(2\pi iab/p)$. There are precisely $2p + 2$ modular invariants: $2p - 2$ of these are automorphism invariants, defined for any $\ell \in \mathbb{Z}_p, \ell \neq 0$, by $\mathcal{Z}_{(a,b),(\ell a, \ell^{-1}b)}^{(\ell)} = 1 = \mathcal{Z}_{(a,b),(\ell b, \ell^{-1}a)}^{(\ell)}$, and all other entries vanish. The remaining modular invariants are $\mathcal{Z}^{m,n}$ for $m, n \in \{0, 1\}$, defined by $\mathcal{Z}_{(ap^m, bp^n), (a'p^n, b'p^m)}^{m,n} = 1$ for all $a, b, a', b' \in \mathbb{Z}_p$ (all other entries are 0). $\mathcal{Z}'^{(\ell)}, \mathcal{Z}^{0,0}, \mathcal{Z}^{1,1}$ are simple-current invariants $\mathcal{Z}_{\langle(\ell,1)\rangle}, \mathcal{Z}_{\langle(1,0)\rangle}, \mathcal{Z}_{\langle(0,1)\rangle}$ respectively.

All of these modular invariants are sufferable. Explicitly, the subgroups H of $\mathbb{Z}_p \times \mathbb{Z}_p$ are $0 \times 0, \langle(1, b)\rangle$ for any $b \in \mathbb{Z}_p, 0 \times \mathbb{Z}_p$, and $\mathbb{Z}_p \times \mathbb{Z}_p$. For $H \simeq \mathbb{Z}_d \times \mathbb{Z}_{d'}$, Künneth (2.1) says $H_H^2(\text{pt}; \mathbb{T}) \simeq \mathbb{Z}_d \otimes_{\mathbb{Z}} \mathbb{Z}_{d'} \simeq \mathbb{Z}_{\gcd(d, d')}$. This means the choice $H = \mathbb{Z}_p \times \mathbb{Z}_p$ has a twist of $\psi \in \mathbb{Z}_p$ but the other $2 + p$ choices for H all come with trivial ψ . The correspondence between \mathcal{Z} and (H, ψ) is given in Table 1.

	(H, ψ)	$K_{H \times G}^0(G \times G)$	neutral	$K_{H \times H}^0(G \times G)$	E	D_+	D_-
$\mathcal{Z}^{(\ell)}$	$(\langle(1, \ell)\rangle, 0)$	$\text{Ver}(G), \mathbb{Z}$ resp for $\ell = 1, \neq 1$	$\text{Ver}(G)$	$\text{Ver}(G)$	$G \times G, 0 \times 0$ resp for $\ell = 1, \neq 1$	0×0	0×0
$\mathcal{Z}'^{(\ell)}$	$(G \times G, \ell)$	R_G	$\text{Ver}(G)$	$R_{G \times G}$	$\langle(\ell, 1)\rangle$	0×0	0×0
$\mathcal{Z}^{0,0}$	$(G \times G, 0)$	R_G	\mathbb{Z}	$R_{G \times G}$	$G \times 0$	$G \times 0$	$G \times 0$
$\mathcal{Z}^{1,1}$	$(0 \times 0, 0)$	$\mathbb{Z}(G)$	\mathbb{Z}	$\mathbb{Z}(G) \otimes \mathbb{Z}(G)$	$0 \times G$	$0 \times G$	$0 \times G$
$\mathcal{Z}^{0,1}$	$(G \times 0, 0)$	\mathbb{Z}	\mathbb{Z}	$R_G \otimes \mathbb{Z}(G)$	0×0	$G \times 0$	$0 \times G$
$\mathcal{Z}^{1,0}$	$(0 \times G, 0)$	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z}(G) \otimes R_G$	0×0	$0 \times G$	$G \times 0$

Table 1. Data for the $G = \mathbb{Z}_p$ modular invariants

The nimrep $K_{H^L \times \Delta_G^R}^*(G \times G) = K_{H^{\text{ad}}}^*(G)$ is recovered as follows. In all cases $K_H^1(G) = 0$; the K -groups $K_H^0(G)$ are collected in Table 1. Note that in all cases $K_{H \times G}^0(G \times G)$ is isomorphic as an additive group to the group ring $\mathbb{Z}(E)$ where the subgroup E of $G \times G$ is defined to be $\{(a, b) \in \mathbb{Z}_p^2 : \mathcal{Z}_{(a,b),(a,b)} = 1\}$. The module structure can be written down as follows. Using the nondegenerate pairing $(a, b) \cdot (a', b') = ab' + a'b \in \mathbb{Z}_p$ introduced earlier, for any subgroup E of $G \times G$ define $E^* = \{(a, b) \in \mathbb{Z}_p^2 : (a, b) \cdot E = \{0\}\}$. Then the boundary states of the nimrep are $\mathcal{B} = (G \times G)/E^*$, which is isomorphic to E as a group, and the primary (a, b) acts on $[x] \in \mathcal{B}$ by addition mod E^* .

The full system $K_{H^L \times H^R}^0(G \times G)$ is collected in Table 1. To identify alpha-induction, it is more convenient to write the full system in equivalent form as

$(G \times G)/D_- \times (G \times G)/D_+^*$, where D_\pm are given in Table 1 and are defined by $D_+ = \{(a, b) : \mathcal{Z}_{(a,b),(0,0)} = 1\}$, $D_- = \{(a, b) : \mathcal{Z}_{(0,0),(a,b)} = 1\}$. Then D_\pm are isomorphic as groups, and both D_\pm^*/D_\pm are isomorphic to the Verlinde ring of the neutral system, also collected in Table 1 ($D_\pm \leq D_\pm^*$ by T -invariance). Alpha-induction is given by $\alpha_{(a,b)}^+ = ((ma, m^{-1}b) + D_-, (a, b) + D_+^*)$ and $\alpha_{(a,b)}^- = ((a, b) + D_-, D_+^*)$ for all \mathcal{Z} (except $\alpha_{(a,b)}^- = ((b, a) + D_-, D_+^*)$ for $\mathcal{Z}'^{(\ell)}$), where $m = \ell$ for $\mathcal{Z}^{(\ell)}$, $m = \ell^{-1}$ for $\mathcal{Z}'^{(\ell)}$, and $m = 1$ for the four other \mathcal{Z} .

More generally, for $G = \mathbb{Z}_{p^\nu}$, the bijection of Table 1 between modular invariants \mathcal{Z} and pairs (H, ψ) is lost. For $G = \mathbb{Z}_{p^\nu}$, there are precisely $((\nu+1)p^{\nu+2} - 2p^{\nu+1} - (\nu+1)p^\nu + 2)/(p-1)^2$ pairs (H, ψ) , each of which is either of the form $H = \langle (\frac{n}{d}, \ell \frac{n}{d}), (0, \frac{n}{d'}) \rangle$ and $\psi \in \mathbb{Z}_{d'}$ for any $d'|d|n$, $0 \leq \ell < \frac{d}{d'}$, or of the form $H = \langle (\ell \frac{pm}{d}, \frac{n}{d}), (\frac{n}{d'}, 0) \rangle$ and $\psi \in \mathbb{Z}_{d'}$ for any $d'|d|n$, $d' < d$, $0 \leq \ell < \frac{d}{d'p}$. $G = \mathbb{Z}_{p^\nu}$ has precisely $2p^\nu - 2p^{\nu-1}$ automorphism invariants: namely $\mathcal{Z}^{(\ell)}$ and $\mathcal{Z}'^{(\ell)}$ for each $\ell \in \mathbb{Z}_{p^\nu}^\times$. These correspond bijectively with the pairs $(\langle (1, \ell) \rangle, 0)$ and $(G \times G, \ell)$ as above. There are precisely $(\nu+1)^2$ $\mathcal{Z}^{m,n}$, defined in the obvious way. They are realised by pairs $(\langle (p^m, 0), (0, p^n) \rangle, 0)$ as above, but for $\nu > 1$ these can be realised by other pairs. For example for $G = \mathbb{Z}_{p^2}$, $\mathcal{Z}^{1,1}$ and $\mathcal{Z}^{0,0}$ are realised by any $(\langle (p, 0), (0, p) \rangle, \mathbb{Z}_p)$ and $(G \times G, p\mathbb{Z}_p)$ respectively. All other of the remaining $(p-1)(p+3)$ modular invariants for $G = \mathbb{Z}_{p^2}$ are also sufferable, although $2(p-1)$ of these are realised by exactly 2 pairs. At the time of writing, we do not know if all modular invariants for cyclic G are sufferable.

3.3 Compact Lie groups

Let $L \subset \mathbb{R}^n$ be any n -dimensional lattice, and write T_L for the tori \mathbb{R}^n/L . Of course all n -tori are homeomorphic, say to T_L , and as explained in [22, Sect.2.2] $\tau \in H_{T_L}^3(T_L; \mathbb{Z})$ -twist here can be taken to be $\tau \in \text{Hom}(L, L^*)$, where $L^* = \{x \in \mathbb{R}^n : x \cdot L \subseteq \mathbb{Z}\}$ is the dual lattice. In this section we will find it convenient to identify these twists τ with their image, i.e. with n -dimensional sublattices M of L^* , so in place of ${}^\tau K_{T_L}^*(T_L)$ we will write $K_{T_L}^*(T_M)$. This K -group was computed in [32, Thm. 3.4.3]:

$${}^\tau K_{T_{L_1}}^i(T_{L_2}) =: K_{T_{L_1}}^i(T_{\tau(L_2)}) \simeq \begin{cases} \mathbb{Z}(L_1^*/\tau(L_2)) & \text{for } i \equiv \dim(T) \pmod{2} \\ 0 & \text{otherwise} \end{cases}. \quad (3.2)$$

All K -groups in this section can be read off from that identity.

We can and will choose L so that the Verlinde ring is the group ring of L^*/L , i.e. is given by $K_{T_L}^{\dim T}(T_L)$. The fusion product is written $[u][v] = [u+v]$; note that each primary, i.e. each coset in L^*/L , is a simple-current. This torus example formally behaves similarly to the case of finite abelian groups considered last subsection, in fact the torus is simple enough that we can work its story out completely.

Consider for simplicity that L is an *even* lattice, i.e. that $u \cdot u \in 2\mathbb{Z}$ for all $u \in L$. Then the matrices $S_{[u],[v]} = |L^*/L|^{-1/2} \exp(2\pi i u \cdot v)$ and $T_{[u],[v]} = \delta_{[u],[v]} \exp(\pi i u \cdot u - \pi i n/12)$ are well-defined, and generate the desired $\text{SL}(2, \mathbb{Z})$ -representation. The vacuum is $[0]$ and charge-conjugation is $C[u] = [-u]$.

Proposition 1(a) *Modular invariants for this chiral data are parametrized by even lattices D_\pm , such that $L \subseteq D_\pm \subseteq L^*$, and an orthogonal isomorphism $\beta : D_+^*/D_+ \rightarrow D_-^*/D_-$ (defined below). The modular invariant is given by*

$$\mathcal{Z}_{[u],[v]} = \begin{cases} 1 & \text{if } u \in D_+^* \text{ and } [v] \subset \beta([u]_+) \\ 0 & \text{otherwise} \end{cases}, \quad (3.3)$$

where we write $[u]_\pm$ for the D_\pm -coset $u + D_\pm$.

(b) *The nimreps (up to equivalence) are parametrised by lattices E , $L \subseteq E \subseteq L^*$: the boundary states $x \in \mathcal{B}$ are the cosets L^*/E^* and the module structure is given by $[u] \cdot [m]_{E^*} = [u + m]_{E^*}$ for all $[u] \in L^*/L$, $[m]_{E^*} \in L^*/E^*$. This nimrep is diagonalised by the matrix $\Psi_{[e],[m]_{E^*}} = |E/L|^{-1/2} \exp(2\pi i e \cdot m)$ for $[e] \in E/L$, $[m]_{E^*} \in L^*/E^*$, i.e. the exponents are the L -cosets E/L , and all multiplicities are 1.*

By *orthogonal isomorphism* we mean a group isomorphism β such that $[u]_+ \cdot [u]_+ \equiv \beta([u]_+) \cdot \beta([u]_+) \pmod{2}$, and hence $[u]_+ \cdot [v]_+ \equiv \beta([u]_+) \cdot \beta([v]_+) \pmod{1}$, for all $[u]_+, [v]_+ \subset D_+^*$. Note that block-diagonal \mathcal{Z} correspond to $D_+ = D_-$ and $\beta = id$, while automorphism invariants correspond to $D_+ = D_- = L$. In particular, the diagonal modular invariant $\mathcal{Z} = I$ is $D_\pm = L$ and $\beta = +1$, while charge-conjugation $\mathcal{Z} = C$ is $D_\pm = L$ with $\beta = -1$. There is a simpler description in the $n = 1$ case, more or less given by [8], but we don't know a similar description in higher dimension. Since the groups E/L and L^*/E^* are isomorphic (Pontryagin duality), the matrix Ψ is square, as it must be.

Proof. Let \mathcal{Z} be any modular invariant. The triangle inequality and $\mathcal{Z} = S\mathcal{Z}S^*$ imply

$$|\mathcal{Z}_{[u],[v]}| = \frac{1}{|L^*/L|} \left| \sum e^{2\pi i u \cdot a} \mathcal{Z}_{[a],[b]} e^{-2\pi i b \cdot v} \right| \leq \frac{1}{|L^*/L|} \sum \mathcal{Z}_{[a],[b]} = \mathcal{Z}_{[0],[0]} = 1,$$

where the sums are over $[a], [b] \in L^*/L$, so each entry $\mathcal{Z}_{[u],[v]}$ either equals 0 or 1. It also says that $\mathcal{Z}_{[u],[v]} = 1$ iff $u \cdot a \equiv v \cdot b \pmod{1}$ whenever $\mathcal{Z}_{[a],[b]} = 1$. Hence we get additivity: if both $\mathcal{Z}_{[u],[v]} = \mathcal{Z}_{[u'],[v']} = 1$, then $\mathcal{Z}_{[u+u'],[v+v']} = 1$.

Define $\mathcal{L} = \cup([u]; [v]) \subset \mathbb{R}^{n,n}$, where the union is over all cosets such that $\mathcal{Z}_{[u],[v]} = 1$, and where we place on it the indefinite inner product $(u; v) \cdot (u'; v') = u \cdot u' - v \cdot v'$. Then \mathcal{L} is an even (by T -invariance) lattice (by additivity). To prove self-duality, suppose $(u_+; u_-) \in \mathcal{L}^*$, i.e. $u_\pm \in L^*$ satisfy $u_+ \cdot u \equiv u_- \cdot v \pmod{1}$ whenever $\mathcal{Z}_{[u],[v]} \neq 0$. Then from $\mathcal{Z} = S\mathcal{Z}S^*$ we get

$$\mathcal{Z}_{[u_+],[u_-]} = \frac{1}{|L^*/L|} \sum e^{2\pi i u_+ \cdot u} \mathcal{Z}_{[u],[v]} e^{-2\pi i u_- \cdot v} = \frac{1}{|L^*/L|} \sum \mathcal{Z}_{[u],[v]} > 0, \quad (3.4)$$

where the sums are over $[u], [v] \in L^*/L$, and thus $(u_+; u_-) \in \mathcal{L}$. Conversely, any even self-dual lattice $\mathcal{L} \subset \mathbb{R}^{n,n}$ containing $(L; L)$ defines a modular invariant: commutation with T is immediate; self-duality requires $\sum_{[u],[v]} \mathcal{Z}_{[u],[v]} = |L^*/L|$ so (3.4) verifies $\mathcal{Z} = S\mathcal{Z}S^*$ at any $(u_+; u_-) \in \mathcal{L} = \mathcal{L}^*$, while if $(u_+; u_-) \notin \mathcal{L}$ then there will be some

$(u; v) \in \mathcal{L}^* = \mathcal{L}$ for which $(u_+; u_-) \cdot (u; v) \notin \mathcal{Z}$ and so $(S\mathcal{Z}S^*)_{[u_+], [u_-]} = 0$ by the usual projection calculation.

Define $D_+ = \{[d] : \mathcal{Z}_{[d], [0]} = 1\}$ and $D_- = \{[d] : \mathcal{Z}_{[0], [d]} = 1\}$. Then these are even lattices satisfying $L \subseteq D_\pm \subseteq L^*$. Because \mathcal{L} is integral, $\mathcal{Z}_{[u], [v]} = 1$ implies both $([u]; [v]) \cdot (D_+; 0) \subseteq \mathbb{Z}$ and $([u]; [v]) \cdot (0; D_-) \subseteq \mathbb{Z}$, i.e. both $u \in D_+^*$ and $v \in D_-^*$. Suppose without loss of generality $|D_+/L| \leq |D_-/L|$. Choose any $[u] \in D_+^*/L$ and compute

$$\begin{aligned} \sum_{[v] \in D_-^*/L} \mathcal{Z}_{[u], [v]} &= \frac{1}{|L^*/L|} \sum_{[a], [b] \in L^*/L} e^{2\pi i u \cdot a} \mathcal{Z}_{[a], [b]} \sum_{[v] \in D_-^*/L} e^{-2\pi i b \cdot v} \\ &= \frac{|D_-^*/L|}{|L^*/L|} \sum_{[a] \in L^*/L, [d] \in D_-/L} e^{2\pi i u \cdot a} \mathcal{Z}_{[a], [d]} = |D_+/L|, \end{aligned} \quad (3.5)$$

using additivity and $|D_-^*/L| = |L^*/D_-|$. But the left-side $\sum_{[v]} \mathcal{Z}_{[u], [v]}$ will be a multiple of $|D_-/L|$, by additivity and the fact that $(0; D_-)$ is a sublattice of \mathcal{L} . Therefore $|D_+/L| = |D_-/L|$ and there is a map $\beta : D_+^*/D_+ \rightarrow D_-^*/D_-$ such that $\beta([d]_+)$ is the unique coset for which $([d]_+; \beta([d]_+)) \in \mathcal{L}$. In fact β is a bijection: if $\beta([d]_+) = \beta([d']_+)$ then $(d - d'; 0) \in \mathcal{L}$ and hence $[d]_+ = [d']_+$. The rest is clear.

Fix a nimrep, with boundary states $\mathcal{B} = \{x_1, \dots, x_N\}$. Then each $[u] \in L^*/L$ permutes the x_i , since it is a simple-current. Note that the indecomposability of the nimrep, required in Definition 2, is equivalent to the transitivity of this permutation representation.

$[e] \in L^*/L$ is an exponent of the nimrep iff there exists a vector $v_{[e]} \neq 0$ (namely the corresponding column of Ψ), with components labelled by x_i , such that $(v_{[e]})_{[u].x_i} = e^{2\pi i e \cdot u} (v_{[e]})_{x_i}$ for all $[u] \in L^*/L$, $x_i \in \mathcal{B}$. Transitivity implies all components of $v_{[e]}$ will be nonzero. If $[e], [e']$ are exponents, then so must be $[e + e']$, whose vector $v_{[e+e']}$ has components $(v_{[e+e']})_{x_i} = (v_{[e]})_{x_i} (v_{[e']})_{x_i}$. Let $E = \cup [e]$ be the union of all exponents; then that additivity means that E is a lattice. Note that the kernel of the nimrep, i.e. the union of all cosets $[u] \in L^*/L$ which act trivially on all x_i , is precisely the dual lattice E^* .

Choose any $u \in L^* \setminus E^*$. Then there exists an exponent $[e'] \in E/L$ such that $u \cdot e' \notin \mathbb{Z}$. Say $[e']$ has order k in L^*/L . Then the permutation matrix $\mathcal{N}_{[u]}$ corresponding to multiplication by $[u]$ has trace

$$\text{tr } \mathcal{N}_{[u]} = \sum_{[e] \in E/L} e^{2\pi i e \cdot u} = \frac{1}{k} \sum_{[e] \in E/L} \sum_{j=0}^{k-1} e^{2\pi i (e + j e') \cdot u} = 0.$$

Therefore any such $[u]$ acts with *no* fixed points. This means the full nimrep can be recovered from say the first column of every $\mathcal{N}_{[u]}$.

For any $[u] \in L^*/L$, $[u].x_1$ depends only on the class $[u] + E^*$ (and x_1), so define $[u].x_1 = x_{[u]+E^*}$. Then this assignment is a bijection between these cosets L^*/E^* and the $x_i \in \mathcal{B}$ (surjectivity is transitivity, and injectivity is fixed-point freeness), satisfying $[u].x_{[u'] + E^*} = x_{[u+u'] + E^*}$. The rest is clear. QED

Theorem 2. Choose any modular invariant $\mathcal{Z}_{[u],[v]}$, given by lattices D_{\pm} and map $\beta : D_+^*/D_+ \rightarrow D_-^*/D_-$. Put $E = \cup_{[u]_+ \in D_+^*/D_+} ([u]_+ \cap \beta([u]_+))$.

(a) E is a lattice, and the nimrep $L^*/E^* = K_{T_L}^{\dim T}(T_{E^*})$ is the only one compatible with the modular invariant. Its $\text{Ver}(T_L)$ -module product $[u].[x]_{E^*} = [u+x]_{E^*}$ is the push-forward of the product in the Verlinde ring $K_{T_L}^{\dim T}(T_L)$.

(b) The full system is $L^*/D_- \times L^*/D_+^* = K_{T_L \times T_L}^0(T_{D_-} \times T_{D_+^*})$, and alpha-induction $\alpha^{\pm} : L^*/L \rightarrow L^*/D_- \times L^*/D_+^*$ is given by $\alpha^+([v]) = (\beta[v]_-, [v]_{+*})$ and $\alpha^-([v]) = ([v]_-, [0]_{+*})$, using obvious notation. The modular invariant is recovered by $\mathcal{Z}_{[u],[v]} = \langle \alpha_{[u]}^+, \alpha_{[v]}^- \rangle$. The neutral system ${}_M\mathcal{X}_M^0$ is $D_-^*/D_- = K_{T_{D_-}}^{\dim T}(T_{D_-})$. Sigma-restriction ${}_M\mathcal{X}_M^0 \rightarrow {}_N\mathcal{X}_N$ is the unravelling $K_{T_{D_-}}^{\dim T}(T_{D_-}) \rightarrow K_{T_L}^{\dim T}(T_{D_-}) \rightarrow K_{T_L}^{\dim T}(T_L)$ of D_- -cosets into L -cosets. The subfactor inclusion ι , its conjugate $\bar{\iota}$, and the canonical endomorphism θ are respectively $[0]_{E^*} \in K_{T_L}^{\dim T}(T_{E^*})$, $[0] \in K_{T_E}^{\dim T}(T_L)$, and $[0]_- = \oplus_{[u] \in D_-} [u]$.

(c) $K^i(T_{E^*}) = \mathbb{Z}^{2^{n-1}}$ for each i , where $n = \dim(T)$. The charge-group \mathcal{M} is \mathbb{Z} , generated by $q_{[x]_{E^*}} = 1$. The map $K_0^{T_L}(T_{E^*}) \rightarrow K_0(T_{E^*})$ forgetting T -equivariance is $1 \otimes_{R_T} K_0^{T_L}(T_{E^*}) \subseteq \mathbb{Z} \otimes_{R_T} K_0^{T_L}(T_{E^*})$, as in section 2.4.

The proof is immediate from the proposition. Because of this uniform K -theoretic description, we would expect all these to be *sufferable*, i.e. realised by subfactors. Note the similarity of Theorem 2 to Table 1 of last subsection, describing the $G = \mathbb{Z}_p$ situation. We see that the full system consists of $|L^*/D_+^*|$ copies of the nimrep $\mathbb{Z}[L^*/D_-]$ for the block-diagonal modular invariant corresponding to the neutral system. In these block-diagonal cases, the neutral system $K_{T_{D_-}}(T_{D_-}) = D_-^*/D_-$ embeds in the nimrep $K_{T_L}(T_{D_-})$, namely through the inclusion $D^*/D = D^*/E^* \subset L^*/E^*$, or K -theoretically through the induction $K_{T_D}^{\dim T}(T_{E^*}) \rightarrow K_{T_L}^{\dim T}(T_{E^*})$. More generally that embedding happens whenever the subfactor inclusion is *type I*.

Incidentally, this faithful parametrisation of modular invariants by lattices \mathcal{L} is formally very similar to the $(H, 0)$ parametrisation for finite groups as discussed in section 3.1: K_{\pm} , N_{\pm} and β there correspond respectively to D_{\pm}^*/L , D_{\pm}/L and β here.

4 Outer automorphisms

4.1 Finite groups

Let G be a finite group and fix any automorphism $\omega \in \text{Aut } G$. It is convenient to introduce the semi-direct product $G_{\omega} = G \rtimes \langle \omega \rangle$, with product $(g, \omega)(g', \omega') = (g\omega(g'), \omega\omega')$. We will see shortly that inner automorphisms are invisible in the K -theoretic description we seek, so really ω lives in $\text{Out } G$.

Note that ω permutes the basis of the Verlinde ring, sending (g, χ) to $(\omega(g), \chi \circ \omega^{-1})$. This permutation defines an automorphism invariant $\mathcal{Z} = I_{\omega}$. We claim that this \mathcal{Z} is sufferable, corresponding to the pair $(H, \psi) = (\Delta_G^{\omega}, 0)$ where Δ_G^{ω} is the ω -twisted diagonal $\{(g, \omega(g)) : g \in G\}$ in $G \times G$.

$K_{\Delta_G^{\omega L \times \Delta_G^{\omega R}}}^*(G \times G) \simeq K_{G^{\text{ad}, \omega}}^*(G)$, with action $g.x = \omega(g)x\omega(g^{-1})$, is obviously isomorphic to $K_{G^{\text{ad}}}^*(G)$. In other words, the full system (and neutral system) is the Verlinde ring. Alpha-induction is clear: α^+ is the action of ω on $\text{Ver}(G)$, while α^- is the identity.

$K_{\Delta_G^{\omega L \times \Delta_G^{\omega R}}}^i(G \times G) \simeq K_{(G,1)^{\text{ad}}}^i((G, \omega))$, regarding both $(G, 1)$ and (G, ω) as subsets of G_ω , is the nimrep for $i = 0$ and vanishes for $i = 1$. Equivalently, this is $K_{G^{\text{ad}, \omega}}^*(G)$ where G acts on itself by the ω -twisted adjoint $g.x = gx\omega(g^{-1})$. The preferred basis \mathcal{B} is obtained automatically from K -theory, namely pairs (g, χ) where now g is a representative of an ω -twisted conjugacy class in G and χ is an irrep of the twisted stabiliser $\mathcal{Z}_g^\omega(G)$.

More generally, one gets graded products (*twisted fusions*) $K_{(G,1)^{\text{ad}}}^0(G, \omega) \times K_{(G,1)^{\text{ad}}}^0(G, \omega') \rightarrow K_{(G,1)^{\text{ad}}}^0(G, \omega' \circ \omega)$, so $\sum_{\omega' \in \langle \omega \rangle} K_{(G,1)^{\text{ad}}}^0(G, \omega')$ is an associative graded ring with unity. All of these products (Verlinde, nimrep, twisted fusion) are simply the push-forward of product on the space G : $g \in G$ acts on $(a, b) \in K_{(G,1)^{\text{ad}}}^0(G, \omega) \times K_{(G,1)^{\text{ad}}}^0(G, \omega')$ by $((\omega^{-1}g)ag^{-1}, gb(\omega'g)^{-1})$, so the product $(a, b) \mapsto ab$ is clearly compatible with this G -action.

We can use ω to twist any modular invariant: $(\mathcal{Z}^\omega)_{\lambda, \mu} = \mathcal{Z}_{\lambda, \omega(\mu)}$. Suppose \mathcal{Z} is sufferable, corresponding to pair (H, ψ) . Then \mathcal{Z}^ω is also sufferable, corresponding to pair $((1, \omega)H, \psi^\omega)$, where $(1, \omega)H = \{(h_1, \omega h_2) : (h_1, h_2) \in H\}$ and where $\psi \mapsto \psi^\omega$ is the isomorphism $H_H^2(\text{pt}; \mathbb{T}) \rightarrow H_{(1, \omega)H}^2(\text{pt}; \mathbb{T})$ obtained from the isomorphism $(1, \omega)H \rightarrow H$.

4.2 Compact Lie groups

The most obvious nondiagonal modular invariant is charge-conjugation $\mathcal{Z} = C$. In the loop group LG setting, charge-conjugation corresponds to an outer automorphism of G . This section develops the K -theoretic interpretation of the nimrep, alpha-induction etc for the modular invariants I_ω associated to any outer automorphism of G (inner automorphisms are invisible). Specialising to the trivial automorphism recovers the Verlinde ring realisation of [30].

Let G be any connected, compact, simply-connected Lie group and fix a level $k \in \mathbb{Z}_{\geq 0}$. Write $\kappa = k + h^\vee$ as in section 2.4. The group of outer automorphisms of G is naturally identified with the group of symmetries of the Dynkin diagram of G , and as a permutation of these vertices also permutes highest weights of G in the usual way and through this the level k primaries $\lambda \in P_+^k(G)$ (ω fixes λ_0). Pick an outer automorphism ω , say of order d , and lift it to an automorphism of G . For example, an automorphism of $G = \text{SU}(n)$ realising charge-conjugation C is complex conjugation.

Introduce the ω -twisted diagonal Δ_G^ω and the semi-direct product $G_\omega = G \rtimes \langle \omega \rangle$ as in section 4.1. As with section 4.1, the full system and alpha-induction are easy here. The full system ${}^\kappa K_{\Delta_G^{\omega L \times \Delta_G^{\omega R}}}^*(G \times G) \simeq {}^\kappa K_{G^{\text{ad}}}^*(G)$ is just the Verlinde ring $\oplus 0$. Implicit here is our identification of the twist groups $H_{\Delta_G^{\omega L \times \Delta_G^{\omega R}}}^3(G \times G; \mathbb{Z}) \simeq H_{G^{\text{ad}}}^3(G; \mathbb{Z})$,

using (2.2) and the spectral sequence of section 2.1. Take α^+ to be ω , and α^- to be the identity. More interesting is the nimrep: it should be ${}^\kappa K_*^{\Delta_G^\omega \times \Delta_G^R}(G \times G) \simeq {}^\kappa K_*^G(G\omega) = {}^\kappa K_*^{G^{\text{ad}_\omega}}(G)$ where ad_ω denotes the ω -twisted adjoint action. Again, spectral sequences permit us to identify these twist groups. For all actions considered in this paragraph, the H^1 -groups vanish. The remainder of this subsection will focus on ${}^\kappa K_*^{G^{\text{ad}_\omega}}(G)$.

The orbits for the untwisted adjoint action of G on G are parametrised by the Stiefel diagram. The analogue for the ω -twisted adjoint action G^{ad_ω} is discussed in [54, 62]. For concreteness consider $G = \text{SU}(3)$ and $\omega = \text{complex conjugation}$; then the twisted Stiefel diagram (see Figure 1(a)) is a segment $0 \leq \theta \leq \pi/4$ with orbit representatives $\text{diag}\left(\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, 1\right)$. The stabiliser at endpoints $\theta = 0$ and $\theta = \pi/4$ are $\text{SO}(3) = \text{Re}(\text{SU}(3))$ and $\text{diag}(\text{SU}(2), 1)$ respectively; at the generic points the stabiliser is $\text{diag}(\text{SO}(2), 1)$.

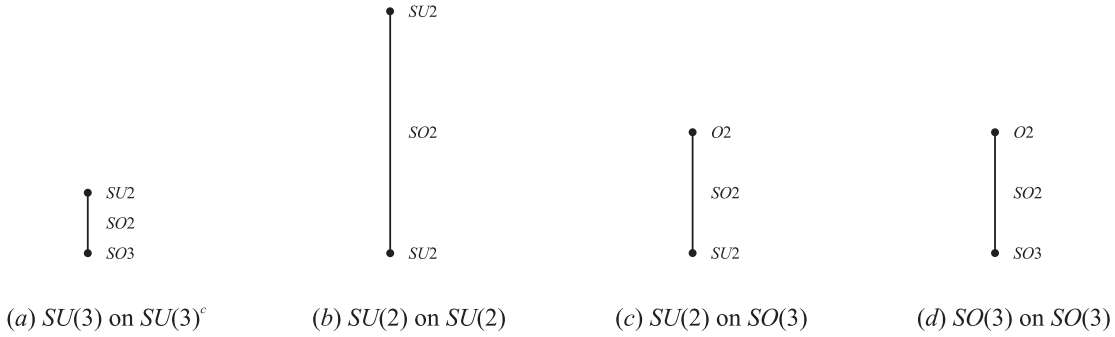


Figure 1. Various one-dimensional orbit diagrams

The Dixmier-Douady bundle realising the H^3 -twist $\kappa \in \mathbb{Z}$ here is constructed exactly like that of G^{ad} on G , using now the twisted Stiefel diagram. In particular, the base is the connected component $G\omega$ of G_ω and the fibres are the compacts $\mathcal{K}(L^2(G\omega) \otimes \ell^2)$. Again consider for concreteness $G = \text{SU}(3)$ and ω being complex conjugation: the level enters through the representation $a \mapsto a^\kappa$ of the stabiliser $\mathbb{T} \simeq \text{SO}(2)$ at the overlap (the midpoint of Figure 1(a)).

Before we describe what happens in general, let's specialise to $G = \text{SU}(3)$ as before. Here $\kappa = k + 3$. From a Mayer-Vietoris calculation we see that $\kappa \in H_G^3(G\omega; \mathbb{Z})$ traces to $(-1)^\kappa \in H_{\text{SO}(3)}^3(\text{SO}(3); \mathbb{Z})$. We can calculate ${}^\kappa K_*^G(G\omega)$ using the 6-term sequence (2.8) by removing the two endpoints of Figure 1(a):

$$\begin{array}{ccccc}
 R_{\text{SO}(2)} & \longleftarrow & {}^\kappa K_0^G(G\omega) & \longleftarrow & 0 \\
 \alpha \downarrow & & & & \uparrow \\
 \mp R_{\text{SO}(3)} \oplus R_{\text{SU}(2)} & \longrightarrow & {}^\kappa K_1^G(G\omega) & \longrightarrow & 0
 \end{array} \tag{4.1}$$

Here \mp denotes $(-1)^{\kappa+1}$; the only subtlety is that Poincaré duality introduces an additional $- \in H_{\text{SO}(3)}^3(\text{pt}; \mathbb{Z})$ twist.

The map α sends $p(a) \in R_{\text{SO}(2)}$ to $\text{D-Ind}_{\text{SO}(2)}^{\text{SO}(3)}(a^{\kappa/2} p, p)$. These Dirac inductions are

given explicitly in [30, App.A]. We find ${}^\kappa K_0^G(G\omega) = 0$ and

$${}^\kappa K_1^G(G\omega) = \begin{cases} R_{\text{SO3}}/(\sigma_\kappa, \sigma_{2i+\kappa} + \sigma_{\kappa-2i})_{0 < i \leq (\kappa-1)/2} & \text{for } \kappa \text{ odd} \\ -R_{\text{SO3}}/(\sigma_\kappa, \sigma_{2i+\kappa} + \sigma_{\kappa-2i})_{0 < i \leq \kappa/2} & \text{for } \kappa \text{ even} \end{cases} \quad (4.2)$$

These K -homology groups naturally inherit an R_G -module structure by restriction (the 3-dimensional fundamental representation $\sigma_{(1,0)} \in R_G$ restricts to $\sigma_3 \in R_{\text{SU2}} \subset R_{\text{SU3}}$) and through this in fact a representation of the fusion ring $\text{Ver}_k(\text{SU3}) = R_G/(\sigma_{(k+1,0)}, \sigma_{(0,k+1)})$: we see by direct calculation that this recovers the appropriate fusion graph (see Figure 2) and hence recovers the nimrep. Indeed, $\sigma_{(1,0)}\sigma_l = (\sigma_2^2 - 1)\sigma_l = \sigma_{l+2} + \sigma_l + \sigma_{l-2}$ for any l .



Figure 2. Nimrep graphs for SU(3) with charge-conjugation

RCFT has a complete (though still conjectural) description for outer automorphism modular invariants ([37], building on [4]), which paints a picture of the modular invariant I_ω for outer automorphisms ω , beautifully parallel to that of the diagonal modular invariant. See Table 2. There $\mathfrak{g}^{(1)}$ is the nontwisted affine algebra of G and $\mathfrak{g}^{(\omega)}$ is the twisted affine algebra of ω -fixed points in $\mathfrak{g}^{(1)}$ [49]. Write $P_+^k(\mathfrak{g}')$ for the level k integrable highest-weights of some affine algebra \mathfrak{g}' . The characters χ_x for $x \in P_+^k(\mathfrak{g}^{(\omega)})$ possess a modularity:

$$\chi_x(-1/\tau) = \sum_{\mu \in P_+^k(\mathfrak{g}^{(\omega)})} S_{x,\mu} \chi_\mu(\tau)$$

for another affine algebra $\mathfrak{g}^{(\omega)}$ called the *orbit Lie algebra*. Ψ in Table 2 refers to the matrix diagonalising the nimrep.

twist ω	\mathcal{B}	exponents	Ψ	nimrep	full system	α^+	α^-
trivial	$P_+^k(\mathfrak{g}^{(1)})$	$P_+^k(\mathfrak{g}^{(1)})$	S for $\mathfrak{g}^{(1)}$	$R_G/I_k(G)$	$\text{Ver}_k(G)$	id	id
nontrivial	$P_+^k(\mathfrak{g}^{(\omega)})$	$P_+^k(\mathfrak{g}^{(\omega)})$	S for $\mathfrak{g}^{(\omega)}$	$R_{G^{(\omega)}}^\pm/I_k^\omega(G)$	$\text{Ver}_k(G)$	ω	id

Table 2. Comparing nontrivial ω twist to trivial twist

Table 3 collects the data for all possibilities for nontrivial ω . x' gives the map $P_+^k(\mathfrak{g}^{(\omega)}) \rightarrow P_+^{k'}(\widehat{G^{(\omega)}})$, where $\widehat{G^{(\omega)}}$ here denotes the universal cover of $G^{(\omega)}$. The twisted fusion ideal $I_k^\omega(G)$ is generated in part by the fusion ideal $I = I_{k'}(G^{(\omega)})$ or its restriction $I^{sp} = I_{k'}(\widehat{G^{(\omega)}}) \cap R_{G^{(\omega)}}^-$ to the spinors. When $x' \pm J'x'$ appears in Table 3, it is meant to include that relation for all x' and all simple-currents J' ; when $x' = J'x'$ then $x' + J'x'$ should be replaced with x' . In the E_6 row, π refers to any of the 5 nontrivial outer automorphisms of D_4 , and ϵ_π its parity; again $x' + \pi x'$ (or $x' + C'x'$) should be replaced with x' when $x' = \pi x'$ (or $x' = C'x'$ respectively). The action of $\text{Ver}_k(G)$ on $R_{G^{(\omega)}}^\pm/I_k^\omega(G)$ is by restriction $R_G \rightarrow R_{G^{(\omega)}}^-$ in all cases, except

for $G = \text{SU}(2n)$ when $G^{(\omega)}$ isn't a subgroup. That case uses the *subjoining* described explicitly in [38], which uses the embeddings $\mathfrak{sp}(2n) \subset \mathfrak{su}(2n)$ and $\mathfrak{sp}(2n) \subset \mathfrak{so}(2n+1)$ to express any G -character restricted to ω -fixed points as a virtual $G^{(\omega)}$ -character. See [37] for any further clarifications on the conventions we use here.

G	ω	$\mathfrak{g}^{(\omega)}$	$\check{\mathfrak{g}}^{(\omega)}$	$R_{G^{(\omega)}}^{\pm}$	k'	$x' \in P_+^{k'}(G^{(\omega)})$	$I_k^{\omega}(G)$
$\text{SU}(2n+1)$	C	$A_{2n}^{(2)}$	$A_{2n}^{(2)}$	$R_{\text{SO}(2n+1)}^-$	$k+2$	$(x_0 + x_1 + 1; x_1, \dots, x_{n-1}, 2x_n + 1)$	$\langle I^{\text{sp}}, x' + J'x' \rangle$
$\text{SU}(2n)$	C	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$	$R_{\text{SO}(2n+1)}^-$	$k+1$	$(x_0; x_1, \dots, x_{n-1}, 2x_n + 1)$	$\langle I^{\text{sp}} \rangle$
$\text{Spin}(2n)$	$\lambda_{n-1} \leftrightarrow \lambda_n$	$D_n^{(2)}$	$A_{2n-3}^{(2)}$	$R_{\text{Spin}(2n-1)}$	$k+1$	$(x_0 + x_1 + 1; x_1, \dots, x_{n-1})$	$\langle I, x' + J'x' \rangle$
$\text{Spin}(8)$	$\lambda_1 \rightarrow \lambda_3 \rightarrow \lambda_4 \rightarrow \lambda_1$	$D_4^{(3)}$	$D_4^{(3)}$	$R_{\text{SU}(3)}$	$k+3$	$(x_0 + x_1 + x_2 + 2; x_2, x_1 + x_2 + 1)$	$\langle I, x' - J'x', x' + C'x' \rangle$
E_6	C	$E_6^{(2)}$	$E_6^{(2)}$	$R_{\text{Spin}(8)}$	$k+6$	$(x_0 + \dots + x_3 + 3; x_1 + x_2 + x_3 + 2, x_4, x_3, x_2 + x_3 + 1)$	$\langle I, x' - J'x', x' - \epsilon_{\pi} \pi x' \rangle$

Table 3. Data for nontrivial ω

[37] identified the weights $P_+^k(\check{\mathfrak{g}}^{(\omega)})$ with the ω -fixed points in $P_+^k(\mathfrak{g}^{(1)}) = P_+^k(G)$, i.e. with the exponents of the desired nimrep. For each $\lambda \in P_+^k(G)$ it defined matrices $\mathcal{N}_{\lambda}^{GG} = \Psi \text{diag}(S_{\lambda, \mu}/S_{1, \mu})_{\mu \in P_+^k(\check{\mathfrak{g}}^{(\omega)})} \Psi^{-1}$ for the unitary matrix Ψ given in the table. Their conjecture is that $\mathcal{N}_{\lambda}^{GG}$ defines a nimrep corresponding to $\mathcal{Z} = I_{\omega}$, and that this nimrep product should correspond to ω -twisted fusions. The boundary states \mathcal{B} then would be identified with $P_+^k(\mathfrak{g}^{(\omega)})$.

Proposition 2. [37, 38, 42] *Let G be a compact, connected, simply-connected Lie group, the level k be any nonnegative integer, and ω any outer automorphism of G . Then the module structure entries $\mathcal{N}_{\lambda, x}^{GG, y}$ for all $\lambda \in P_+^k(G)$ and all $x, y \in P_+^k(\mathfrak{g}^{(\omega)})$ are integers, and in particular coincide with the action of $\text{Ver}_k(G)$ on $R_{G^{(\omega)}}^{\pm}/I_k^{\omega}(G)$, with respect to the basis x' , defined above. If all $\mathcal{N}_{\lambda, x}^{GG, y}$ are in fact nonnegative, then $\mathcal{N}_{\lambda}^{GG}$ defines a nimrep corresponding to $\mathcal{Z} = I_{\omega}$. Assuming Conjecture 1 of section 2.4, the map $I_k^{\omega}(G) \rightarrow \mathbb{Z}$ given by dimension of the $G^{(\omega)}$ virtual representation, is surjective onto $M_k(G)\mathbb{Z}$; the charge-group $\mathcal{M}_{\mathcal{N}^{GG}}$ is $\mathbb{Z}_{M_k(G)}$, and is generated by the charge assignment $q_x = \dim(x')$.*

Does the proposed K -homology for the nimrep match the conjectured \mathcal{N}^{GG} ? Can K -theoretic methods be used to prove the nonnegativity needed in Proposition 2? Can K -theoretic considerations explain some of the arbitrariness in the RCFT description, in particular in the choice of $G^{(\omega)}$ and the presence of subjoinings for $G = \text{SU}(2n)$?

Theorem 3. *Let G be any compact, connected, simply-connected Lie group, any level $k \in \mathbb{Z}_{\geq 0}$, and ω any automorphism of G . Write $\kappa = k + h^{\vee}$ as usual. Then ${}^{\kappa}K_{1+\dim G+\dim G^{(\omega)}}^{G^{\text{ad}}}(G\omega) = 0$. A natural basis for ${}^{\kappa}K_*^G(G\omega)$ can be identified with $P_+^k(\mathfrak{g}^{(\omega)})$. We get a natural product $\text{Ver}_k(G) \times {}^{\kappa}K_*^G(G\omega) \rightarrow K_*^G(G\omega)$, written say $[\lambda] * [x] = \sum_{[y]} \mathcal{N}_{\lambda, [x]}^{K^{th} [y]} [y]$. These coefficients $\mathcal{N}_{\lambda, [x]}^{K^{th} [y]}$ are nonnegative integers, and the corresponding matrices $\mathcal{N}_{\lambda}^{K^{th}}$ form a representation of $\text{Ver}_k(G)$. The groups $H_{G^{\text{ad}}}^3(G\omega; \mathbb{Z})$ and $H^3(G; \mathbb{Z})$ are naturally identified.*

The evaluation of this K -homology is from Theorem 1 above. The appearance of $P_+^k(\mathfrak{g}^{(\omega)})$ exactly matches [37]. Section 16 of [33] relates this product $\text{Ver}_k(G) \times {}^\kappa K_G^*(G^{\text{ad}_\omega}) \rightarrow K_G^*(G^{\text{ad}_\omega})$ to Segal's fusion, hence the multiplicities $\mathcal{N}_{\lambda, [x]}^{Kth [y]} [y]$ are nonnegative integers. This makes it easy to conjecture the following.

Recall the exterior algebra $\bigwedge\{x_1, \dots, x_{r-1}\}$ from section 2.4. We can rewrite this as $\bigwedge\{x_1, \dots, x_r\}/(x_1 + \dots + x_r)$; from the argument of [17] the x_i are associated with the fundamental weights of G . To any n -dimensional sublattice L of the weight lattice $P = \text{span}_{\mathbb{Z}}\{x_1, \dots, x_r\}$, we obtain a unique vector v_L (well-defined up to ± 1) in $\bigwedge^n\{x_1, \dots, x_r\}$ by taking any basis y_1, \dots, y_n of L and forming the wedge product $y_1 \wedge \dots \wedge y_n$.

Conjecture 2(a) *The action of $\text{Ver}_k(G)$ on ${}^\kappa K_{\dim G + \dim G^{(\omega)}}^{G^{\text{ad}}}(G\omega)$ is equivalent to the conjectured nimrep \mathcal{N}^{GG} of [37], corresponding to the modular invariant $\mathcal{Z} = I_\omega$.*

(b) *The D -brane charges for the nimrep ${}^\kappa K_*^{G^{\text{ad}}}(G\omega)$ is the natural map $q_\omega : {}^\kappa K_{\dim G + \dim G^{(\omega)}}^G(G\omega) \rightarrow {}^\kappa K_{\dim G + \dim G^{(\omega)}}(G)$, forgetting G -equivariance. It is defined by $q_\omega([\bar{\lambda}]) = (\dim(\bar{\lambda}) \pmod{M})[v_L]$ where L is the sublattice $(L^\omega)^\perp$ of the weight lattice P orthogonal to the fixed-points L^ω , and $[v_L]$ means the corresponding element in $\bigwedge^{\dim G - \dim G^{(\omega)}}\{x_1, \dots, x_r\}/(x_1 + \dots + x_r)$. This charge can also be defined by the commuting diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow & I_k^\omega(G) & \longrightarrow & R_{G^{(\omega)}}^\pm = K_0^{G^{(\omega)}}(1) & \xrightarrow{\alpha} & {}^\kappa K_0^G(G\omega) & \longrightarrow 0 \\ & \beta \downarrow & & \beta \downarrow & & \gamma \downarrow & \\ 0 \longrightarrow & M\mathbb{Z} & \longrightarrow & \mathbb{Z} = K_0(1) & \xrightarrow{\delta} & {}^\kappa K_0(G) & \end{array}, \quad (4.3)$$

where β is dimension, and α is defined by a combination of inducing $K_* \rightarrow K_*^G$ and the push-forward of the inclusion $1 \hookrightarrow G$.

Being $\text{Ver}_k(G)$ -modules, \mathcal{N}_λ^{GG} and $\mathcal{N}_\lambda^{Kth}$ are both uniquely determined by their values on the fundamental weights $\Lambda_1, \dots, \Lambda_r$, since the Verlinde ring is a homomorphic image of the polynomial ring on those fundamentals. Together with Theorem 3 then, we can establish the equality $\mathcal{N}_\lambda^{GG} = \mathcal{N}_\lambda^{Kth}$ for all λ , and the nimrep property, both conjectured in Conjecture 2(a), if we can establish that $\mathcal{N}_{\Lambda_i}^{GG} = \mathcal{N}_{\Lambda_i}^{Kth}$ for all $1 \leq i \leq r$. This is perhaps the most promising way to establish the nimrep property for \mathcal{N}_λ^{GG} . These coefficients $\mathcal{N}_{\Lambda_i}^{GG}$ can be deduced from Proposition 2, or found explicitly in [37, 38, 42]: for example, for $\text{SU}(2n+1)$ we have $\mathcal{N}_{\Lambda_i, x}^{GG y} = \mathcal{N}_{\Lambda_{2n+1-i}, x}^{GG y} = N_{\Lambda'_i, x'}^{y'}$ for $1 \leq i < n$ and $\mathcal{N}_{\Lambda_n, x}^{GG y} = \mathcal{N}_{\Lambda_{n+1}, x}^{GG y} = N_{2\Lambda'_i, x'}^{y'}$, where primes denote the fundamental weights and fusions of $\text{SO}(2n+1)$ level $k+2$.

It should be possible to compute ${}^\kappa K_{G^{\text{ad}}}^*(G\omega)$ and the forgetful map ${}^\kappa K_{G^{\text{ad}}}^*(G\omega) \rightarrow {}^\kappa K^*(G)$ by generalising the methods of [52] to nontrivial ω using the twisted Stiefel diagrams of [54], and then applying the Hodgkin spectral sequence. This would go a long way toward proving Conjecture 2.

Because of this uniform K -theoretic description, we would expect all of these modular invariants I_ω to be sufferable. Along these lines, Verrill [65] (with a gap filled

in [69]) does for the C -twisted loop group associated to $SU(2n)$ what Wassermann et al (see e.g. [68]) did for the nontwisted loop groups, and realised the twisted fusions using subfactors.

Any modular invariant \mathcal{Z} can be twisted by an outer automorphism, by multiplying \mathcal{Z} by the automorphism invariant I_ω . Moreover, in the subfactor approach the matrix product of sufferable modular invariants will itself be sufferable. It is tempting then to guess one can always apply the outer automorphism ω to any K -homological description of the data associated to a modular invariant \mathcal{Z} , to get a K -homological description of $I_\omega \mathcal{Z}$ or $\mathcal{Z} I_\omega$. This was proved in the finite group setting at the end of section 4.1; the most important example of this in the loop group setting is given in section 5.3.

5 Simple-current modular invariants

5.1 Finite groups

This section addresses the simple-current modular invariants, also called the D-series or simple-current orbifold invariants. As usual consider first a finite group G . Its simple-currents consist of primaries (z, ϕ) where $z \in Z(G)$ and $\phi \in \widehat{G}$ is dimension-1. Any simple-current of a finite group gives rise to a modular invariant by (2.10) (this fails for e.g. the odd levels of $SU(2)$); it will be an automorphism invariant iff the root of unity $\phi(z)$ has order equal to the least common multiple of the orders of ϕ in \widehat{G} and z in $Z(G)$. Incidentally, most finite groups have simple-currents; the only one of order < 168 which doesn't is the alternating group A_5 .

Consider for concreteness $G = D_{2n} := \langle r, s : r^2 = s^{2n} = rsrs = 1 \rangle$, the dihedral group with $4n$ elements. Let ψ_{ij} , $i, j \in \{0, 1\}$, denote its 4 1-dimensional irreps, defined by $\psi_{ij}(r^a s^b) = (-1)^{ia+jb}$; denote the remaining $n-1$ 2-dimensional irreps by χ_k , $1 \leq k < n$ using obvious notation. Its 8 simple-currents are $z_{hij} := (s^{hn}, \psi_{ij})$ for all $h, i, j \in \mathbb{Z}_2$; apart from the trivial z_{000} , they all have order 2. The remaining $2n^2+6$ primaries are (s^{hn}, χ_k) , (s^a, ϕ_l) , (rs^h, ψ'_{ij}) where $1 \leq a < n$, $l \in \mathbb{Z}_{2n}$, $h, i, j \in \mathbb{Z}_2$, where $\psi'_{ij}((rs^h)^a s^{bn}) = (-1)^{ai+bj}$ etc. See section 3.2 of [15] for more details, as well as the S and T matrices.

Recall from section 2.3 that a simple-current j permutes the primaries, and that it associates to each primary μ a rational number $Q_j(\mu)$. The simple-current z_{hij} is order-2 (unless $h = i = j = 0$), so $2Q_{z_{hij}}(\mu)$ is an integer which we'll call the parity. The permutations and parities for z_{hij} are: $(s^{nh'}, \psi_{i'j'}) \mapsto (s^{n(h+h')}, \psi_{i+i', j+j'})$ (with parity $n(jh' + j'h)$), $(s^{nh'}, \chi_k) \mapsto (s^{n(h+h')}, \chi_{nj+(-1)^{j_k}})$ (parity $hk + nh'j$), $(s^a, \phi_l) \mapsto (s^{nh+(-1)^{h_a}}, \phi_{(-1)^{h_l+n_j}})$ (parity $ja + hl$), and $(rs^{h'}, \psi'_{i'j'}) \mapsto (rs^{\{h'+nh\}_2}, \psi'_{i'+i+h'j, j'+n_j})$ (parity $j'h + i + h'j$), where $\{k\}_2 \in \{0, 1\}$ is congruent mod 2 to k . The only fixed-points of nontrivial z_{hij} are $(s^{h'n}, \chi_k)$ and (s^a, ϕ_l) for z_{010} , and (when n is even) $(rs^{h'}, \psi'_{i'j'})$ for both z_{100} and $z_{hh'1}$, $(s^{nh'}, \chi_{n/2})$ for z_{0i1} , $(s^{n/2}, \phi_{\pm n/2})$ for z_{1i1} , and both $(s^{n/2}, \phi_0)$ and $(s^{n/2}, \phi_n)$ for z_{1i0} .

Write $\mathcal{Z}^{(hij)}$ for the simple-current invariant $\mathcal{Z}_{\langle z_{hij} \rangle}$. Among the $\mathcal{Z}^{(hij)}$, the only

nontrivial automorphism invariants occur when n is odd with $h = j = 1$. These automorphism invariants permute the primaries, fixing those with even parity and interchanging $\lambda \leftrightarrow z_{1i1}\lambda$ for λ with odd parity. The other $\mathcal{Z}^{(hij)}$ are all block-diagonal modular invariants: their only nonzero entries are $\mathcal{Z}_{\lambda,\lambda}^{(hij)} = 2$ when λ is a fixed-point of z_{hij} with even parity, and $\mathcal{Z}_{\lambda,\lambda}^{(hij)} = \mathcal{Z}_{\lambda,z_{hij}\lambda}^{(hij)} = 1$ when λ has even parity but is not a fixed-point. In addition, products of any two distinct $\mathcal{Z}^{(hij)}$ is a new modular invariant.

To keep things simple, we will restrict here to the case of direct relevance to the loop group setting (our primary interest in this paper). Take $H = \langle (b^n, 1), (a, a), (b, b) \rangle \simeq Z \times G$ for $Z \simeq \mathbb{Z}_2$ the centre of D_{2n} . Then by Künneth $H_H^2(\text{pt}; \mathbb{T}) \simeq \mathbb{Z}_2^4$, where one copy of \mathbb{Z}_2 comes from $H_{D_{2n}}^2(\text{pt}; \mathbb{T})$, and the other three come from the characters of Z and G . Again, we will restrict to trivial $\psi \in H_H^2(\text{pt}; \mathbb{T})$ although it should be clear how to modify this discussion for non-trivial ψ . In this case the nimrep is $K_{H^L \times \Delta_G^R}^0(G \times G) \simeq K_{Z^L \times G^{\text{ad}}}^0(G) \simeq K_{G^{\text{ad}}}^0(G/Z)$, with basis given by pairs (Za, χ) for any irrep χ of $Z_{Za}(G)$. The full system is $K_{H^L \times H^R}^0(G \times G) \simeq R_Z \otimes K_{G^{\text{ad}}}^0(G/Z)$, i.e. two copies of the nimrep. Then alpha-induction $\alpha^\pm : K_{G^{\text{ad}}}^0(G) \rightarrow K_{G^{\text{ad}}}^0(G/Z)$ is the map $(a, \chi) \mapsto (Za, \text{Ind}_{Za}^{Z_{Za}} \chi)$ and $(a, \chi) \mapsto (Za, \text{Ind}_{Za}^{Z_{Za}} \chi \circ s^n)$ respectively (i.e. for α^- premultiply and then induce). We find that the corresponding modular invariant is $\mathcal{Z}^{(100)}$. Next subsection we find that a similar description applies to loop groups.

5.2 Compact Lie groups

Simple-current modular invariants are the generic modular invariants for the loop groups, and their geometry is clear: they correspond to strings living on non-simply-connected groups G/Z where Z is some subgroup of the centre of G . These are related to section 3, except fixed points of the simple-currents occur here, complicating things considerably.

The simple-currents in $\text{Ver}_k(G)$ for any connected, simply-connected, compact Lie group G , were classified in [35]. The group of simple-currents for $G \times H$ is the direct product of those for G and H , so it suffices to consider simple G . All of these simple-currents correspond to extended Dynkin diagram symmetries, with one exception ($\text{Ver}_2(E_8)$) which we can ignore as it cannot yield a modular invariant for G . For any G and k , the simple-currents and outer automorphisms together generate all symmetries of the extended Dynkin diagram.

For example, the group of simple-currents for $\text{Ver}_k(\text{SU}(n))$ is cyclic of order n , generated by $J = (0; k, 0, \dots, 0)$ which permutes $P_+^k(\text{SU}(n))$ through $(\lambda_0; \lambda_1, \dots, \lambda_{n-1}) \mapsto (\lambda_{n-1}; \lambda_0, \lambda_1, \dots, \lambda_{n-2})$. Then $Q_{J^d}(\lambda) = \frac{d}{n} \sum_{i=1}^{n-1} i \lambda_i$ and $h_{J^d} = kd(n-d)/2n$.

First, we need to establish K -theoretically the relation, well appreciated within conformal field theory (see e.g. [28]), between the simple-currents and the centre of G . Consider for concreteness $G = \text{SU}(2)$. Recall the Dixmier-Douady bundle for G on G constructed in section 2.2, and let S denote the Stiefel diagram, i.e. $\text{diag}(e^{i\theta}, e^{-i\theta})$ for $0 \leq \theta \leq \pi$. For $x \in S$ and compact $c \in \mathcal{K}$, define the action of $z = -I \in Z(G)$

on fibres by $(gxg^{-1}, c)_1 \leftrightarrow (zgxg^{-1}, c)_2$ for all $g \in G$. G -equivariance is automatic, as is the consistency condition for $g \in Z_x(G)$ when $x \in D_1 \cap D_2$, where D_i is the cover of S . To see that this action preserves the twist, note that $U_k \pi_w U_k^* = 1^{-k} \pi_w$ where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ($\text{Ad}(w)$ moves zx back into the Stiefel diagram, and sends weightspace $V_{m,n-1}$ to ${}^n V_{-m,1-n}$). Therefore the full centre $\pm I$ of $G = \text{SU}(2)$ acts on the G on G bundle, and hence on the K -group ${}^\kappa K_{G^{\text{ad}}}^1(G)$.

We expect that the same conclusion should apply to the centre of any simply-connected, connected, compact Lie group G . For such G , multiplication by the centre $Z(G)$ should correspond naturally to the action of the simple-currents in the Verlinde ring ${}^\kappa K_{G^{\text{ad}}}^{\dim G}(G)$, in the following sense. The primaries $\lambda \in P_+^k(G)$ are identified with certain conjugacy classes — this yields a geometric picture of $\text{Ver}_k(G)$ dual to the usual representation ring description R_G/I_k . Now, $Z(G)$ permutes these conjugacy classes by multiplication, and this permutation agrees with the simple-current action on primaries. For example, for $G = \text{SU}(2)$, the level k primaries look like $\lambda = (k - \lambda_1; \lambda_1)$ for integers $0 \leq \lambda_1 \leq k$; this corresponds to the conjugacy class intersecting S at $\text{diag}(\exp[2\pi i(\lambda_1 + 1)/2\kappa], \exp[-2\pi i(\lambda_1 + 1)/2\kappa])$ for $\kappa = k + 2$. By multiplication, the central element $z = -I$ sends the λ_1 conjugacy class to the $\lambda_1 + \kappa$ one, which the Weyl group identifies with $\kappa - \lambda_1 - 2$. This matches the action of the simple-current.

We can also see this identification between $Z(G)$ and simple-currents, in the representation ring picture. Again consider for concreteness $G = \text{SU}(2)$: we can compute its Verlinde ring $\text{Ver}_k(G) = {}^\kappa K_0^G(G)$ using the six-term sequence (2.8) by removing three points from Figure 1(b) (the 2 endpoints and the midpoint):

$$\begin{array}{ccccc} 0 & \longleftarrow & {}^\kappa K_0^G(G) & \longleftarrow & R_G \oplus R_T \oplus R_G \\ & & \downarrow & & \uparrow \beta \\ 0 & \longrightarrow & {}^\kappa K_1^G(G) & \longrightarrow & R_T \oplus R_T \end{array}, \quad (5.1)$$

where T is the maximal torus. The map β sends the polynomials $(p(a), q(a)) \in R_T^2$ to Dirac induction $(\text{D-Ind}_T^G p, p + a^{-\kappa} q, \text{D-Ind}_T^G q)$, and we obtain ${}^\kappa K_1^G(G) = 0$ and

$$\text{Ver}_k(G) = {}^\kappa K_0^G(G) = R_G^2 / \text{span}\{(\sigma_i, \sigma_{i+\kappa})\}_{i \in \mathbb{Z}}. \quad (5.2)$$

Hence $\text{Ver}_k(G)$ is spanned by the classes $[(0, \sigma_l)]$ for $1 \leq l < \kappa$. The centre $Z = \{\pm I\}$ permutes the two R_G 's in the upper-right corner of (5.1), i.e. in $K_G^0(Z) \simeq R_G^2$, and so sends $[(0, \sigma_l)]$ to $[(\sigma_l, 0)] = [(0, \sigma_{\kappa-l})]$. Again, this permutation matches the simple-current action in $\text{Ver}_k(G)$.

For any subgroup Z of the centre $Z(G)$ of any connected, simply-connected, compact G , R_Z is an R_G -module with action $\rho_\lambda \cdot \psi = \overline{\text{Res}}_Z^G(\rho_\lambda) \psi$, where $\overline{\text{Res}}_Z^G(\rho_\lambda)$ denotes the unique 1-dimensional representation appearing in the restriction of the G -irrep ρ_λ to the central subgroup Z . The same action defines a $\text{Ver}_k(G)$ -module structure, and indeed a $\text{Ver}_k(G)$ -nimrep, on R_Z for any level k . The exponents of this nimrep are the simple-currents corresponding to Z (each with multiplicity 1). Moreover, for any $z \in Z$ and $\lambda \in P_+^k(G)$, $\overline{\text{Res}}_Z^G(\rho_\lambda)(z) = e^{2\pi i Q_j(\lambda)}$, where j is the simple-current

corresponding to z and Q_j is defined early in section 2.3. This nimrep R_Z will generally not have a compatible modular invariant, but it arises indirectly in both this subsection and the next.

Now that we have identified the simple-currents with the centre, we can express K -theoretically the nimreps, full systems, etc for the simple-current modular invariants. We will do this for any $G = \mathrm{SU}(n)$ in this subsection, and comment briefly at the end of the subsection on the related description for other G .

Consider first $G = \mathrm{SU}(2)$ for concreteness. Write $Z = \{\pm I\}$ for its centre and $\overline{G} = G/Z \simeq \mathrm{SO}(3)$. Let k be any nonnegative *even* integer — we require k even here for the existence of the simple-current invariant $\mathcal{Z}_{\langle J \rangle}$, nimrep etc (more on this shortly). Write $\kappa = k+2$ as usual. The Verlinde ring is described by the (unextended) \mathbb{A}_{k+1} diagram, and the nimrep by the $\mathbb{D}_{k/2+2}$ diagram, as explained in section 2.3. Apart from this, the theory bifurcates between $k/2$ even (where $\mathcal{Z}_{\langle J \rangle}$ is block-diagonal) and $k/2$ odd (where it is an automorphism invariant).

For later convenience, we can redo the $\mathrm{Ver}_k(\mathrm{SU}(2))$ calculation of (5.1) (for κ even), taking $\beta = (\mathrm{D}\text{-}\mathrm{Ind}_T^G p, a^{\kappa/2}p + a^{-\kappa/2}q, \mathrm{D}\text{-}\mathrm{Ind}_T^G q)$ and obtaining the equivalent expression for $\mathrm{Ver}_k(\mathrm{SU}(2))$:

$${}^\kappa K_0^G(G) = R_T / \mathrm{span}\{a^{\kappa/2}, a^{-\kappa/2}, a^{i+\kappa/2} + a^{-i+\kappa/2}, a^{j-\kappa/2} + a^{-j-\kappa/2}\}_{i,j \geq 1}. \quad (5.3)$$

The obvious \mathbb{Z} -basis of $\mathrm{Ver}_k(G)$ is $[a^{-\kappa/2+1}], \dots, [1], \dots, [a^{\kappa/2-1}]$. The R_G action inherited from K -theory is by restriction to T , so for example the generator σ_2 acts by multiplication by $a + a^{-1}$. In terms of the given basis, multiplication by σ_2 recovers the $\mathbb{A}_{2\kappa-1}$ diagram. Therefore the R_G -module product descends to the Verlinde ring product.

The nimrep corresponds to ${}^\tau K_0^G(\overline{G}) \simeq {}^\tau K_0^{G^{\mathrm{ad}} \times Z^R}(G)$. This K -homology has already been computed in [12], but it is convenient to recompute it in order to compare below various K -groups. In addition, our calculation is far simpler. The twist τ here belongs to $H_G^1(\overline{G}; \mathbb{Z}_2) \times H_G^3(\overline{G}; \mathbb{Z}) \simeq \mathbb{Z}_2 \times \mathbb{Z}$ by (2.3). The spectral sequence of section 2.1 identifies $H_G^3(G; \mathbb{Z})$ with $H^3(G; \mathbb{Z}) \simeq \mathbb{Z}$, and $H_G^3(\overline{G}; \mathbb{Z})$ with $2\mathbb{Z} \subset H^3(G; \mathbb{Z})$. This factor of 2 plays a crucial role below.

(2.2) naturally identifies $H_{G^{\mathrm{ad}} \times Z^L}^*(G; A)$ with $H_G^*(\overline{G}; A)$ and thus it suffices to construct the Dixmier-Douady bundle for ${}^\tau K_{G^{\mathrm{ad}} \times Z^L}^*(G)$. Focus for now on the H^3 -twist. Recall the bundle for ${}^\kappa K_G^*(G)$ given in section 2.2, and use the action of Z on it (which works for any κ) to try to enhance the G -equivariance to $G \times Z$. The obstruction is $(-1)^\kappa$, because the forgetful map $H_{G \times Z}^3(G; \mathbb{Z}) \rightarrow H_G^3(G; \mathbb{Z})$ is $\times 2$. To see this topologically, observe that the orbit diagram for $G \times Z$ on G is given by that of Figure 1(c), with centralisers G , the maximal torus T of G , and $T \times Z$. In particular, the diagram is folded in 2 from that of G on G (Figure 1(b)). Let κ' be the level of $G \times Z$ on G ; i.e. the twisting unitary in the bundle is the representation $a^{\kappa'}$ of the centraliser T . The corresponding bundle for G on G , forgetting the Z action, unfolds the orbit diagram, giving *two* cuts, each with a twisting unitary given by the same representation $a^{\kappa'}$ of the same centraliser T . This is equivalent of course to a single cut, with a twist of $2\kappa'$.

To incorporate an H^1 -twist on this bundle, split $L^2(G) = \mathcal{H}_{ns} \oplus \mathcal{H}_{sp}$ into non-spinors and spinors — this defines a \mathbb{Z}_2 -grading on the space — and use the odd automorphism $U = \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}$ (see section 2.2 of [22] for a similar construction).

Restrict for now to the twist $\tau = (-; \kappa/2)$ relevant to the modular invariant $\mathcal{Z}_{\langle J \rangle}$ — we use a semi-colon to separate the H^1 and H^3 components. The $-$ here arises from the adjoint shift, as does the $+1$ in the H^3 -component. The orbits of G on \overline{G} are given in Figure 1(c). Now use the six-term sequence (2.8), removing the two endpoints of Figure 1(c):

$$\begin{array}{ccccc} 0 & \longleftarrow & (-; \kappa/2) K_0^G(\overline{G}) & \longleftarrow & R_G \oplus {}^+ R_{O_2} \\ \downarrow & & & & \uparrow \alpha \\ 0 & \longrightarrow & (-; \kappa/2) K_1^G(\overline{G}) & \longrightarrow & R_T \end{array}, \quad (5.4)$$

where we have used the nonorientability of the projective plane $G/O(2)$ to obtain the ungraded representation ring R_{O_2} (recall the implicit use of Poincaré duality (2.4) here). The map $\alpha : R_T \rightarrow R_G \oplus R_{O_2}$ sends p to $(\text{D-Ind}_T^G p, \text{Ind}_T^{O_2} a^{\kappa/2} p)$. We find (for $\kappa \neq 0$) that $(-; \kappa/2) K_1^{\text{SU}^2}(\text{SO}3) = 0$ and

$$(-; \kappa/2) K_0^{\text{SU}^2}(\text{SO}3) = R_{O_2} / \text{span}\{\kappa_{\kappa/2}, \kappa_{j+\kappa/2} + \kappa_{-j+\kappa/2}\}_{j \geq 1}. \quad (5.5)$$

The R_G -module structure arising from K -homology is restriction to $O(2)$ from G . For example, the generator σ_2 restricts to κ_1 ; in terms of the obvious \mathbb{Z} -basis of ${}^\tau K_0^G(\overline{G})$, namely $[\delta], [\kappa_0], \dots, [\kappa_{k/2}]$, we obtain the (unextended) Dynkin diagram $\mathbb{D}_{\kappa/2+1} = \mathbb{D}_{k/2+2}$. We see that this R_G -module multiplication factors through the fusion ideal and is thus a well-defined action of $\text{Ver}_k(G)$, recovering precisely the nimrep. A different expression for this nimrep, generalising to other $\text{SU}(n)$, is obtained in Theorem 4 below.

When $k/2$ is odd, the modular invariant $\mathcal{Z}_{\langle J \rangle}$ is an automorphism invariant and the full system is simply the Verlinde ring, as is the neutral system. Alpha-induction $\alpha^\pm : \text{Ver}_k(G) \rightarrow \text{Ver}_k(G)$ then is given by $\alpha^- = \text{id}$ and $\alpha^+([\sigma_i]) = [\sigma_i]$ or $[\sigma_{\kappa-i}]$ for i odd/even respectively, recovering $\mathcal{Z}_{\langle J \rangle}$.

Much more interesting is $k/2$ even, where $\mathcal{Z}_{\langle J \rangle}$ is block-diagonal. The neutral system is given by ${}^\tau K_0^{\overline{G}}(\overline{G})$ where $\tau = (-; +, \kappa/2) \in \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z})$. Again, the spectral sequence tells us the map from $H_G^3(G; \mathbb{Z}) \simeq \mathbb{Z}$ to $H_{\overline{G}}^3(\overline{G}; \mathbb{Z}) \simeq \mathbb{Z}_2 \times \mathbb{Z}$ is by multiplication by 2 in the second component. This K -homology is computed explicitly in [30] but again it is convenient to recalculate it in a slightly different way. The orbit diagram is Figure 1(d); using (2.8) recovers the same sequence as in [30], with the map $\gamma : R_T^- \rightarrow R_{\text{SO}3} \oplus R_{O_2}$ sending the spinor $a^{1/2}p(a)$ to $(\text{D-Ind}_T^{\text{SO}3} a p(a^2), \text{Ind}_T^{O_2} a^{\kappa/4} a^{1/2} p(a))$. The grading on R_{O_2} is lost by the implicit application of Poincaré duality. We find $(-; +, \kappa/2) K_1^{\text{SO}3}(\text{SO}3) = 0$ and

$$\begin{aligned} (-; +, \kappa/2) K_0^{\text{SO}3}(\text{SO}3) &= \text{Ver}_{k/2}(\text{SO}(3)) \\ &= R_{O_2} / \text{span}\{\overline{\kappa}_{\kappa/4+i} + \overline{\kappa}_{\kappa/4-i}\}_{i=1/2, 3/2, \dots}. \end{aligned} \quad (5.6)$$

The appropriate basis is $[\bar{\delta}], [\bar{\kappa}_0], \dots, [\bar{\kappa}_{k/4}]$, which recovers the extended fusions of $\text{SO}(3)$ at $\text{SO}(3)$ -level $k/2$. This answer is different from, but equivalent to, that given in [30]. The bar's atop the representations here emphasise that this $\text{O}(2)$ lies in $\text{SO}(3)$.

In pure extension or type I theories such as these, the neutral system should embed into the nimrep. To recover this here, compare the corresponding six-term sequences term by term: we find that the map $(-; +, \kappa/2) K_0^{\bar{G}}(\bar{G}) \rightarrow (-; \kappa/2) K_0^G(\bar{G})$ is the obvious injection $[\bar{\delta}] \mapsto [\delta], [\bar{\kappa}_i] \mapsto [\kappa_{2i}]$.

The full system for $k/2$ even is ${}^{\tau'} K_0^{\Delta_G^L \times \Delta_G^R}(\bar{G} \times \bar{G}) \simeq {}^{\tau'} K_0^{G^{\text{ad}} \times Z^L \times Z^R}(G) \simeq R_Z \otimes_{\mathbb{Z}} {}^{\tau''} K_0^{G \times Z}(G)$. The twist groups are $H_{G \times Z \times Z}^1(G; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H_{G \times Z \times Z}^3(G; \mathbb{Z}) \simeq 2\mathbb{Z} \times 2\mathbb{Z}$, calculated as before. The twist relevant to $\mathcal{Z}_{\langle J \rangle}$ is $\tau' = (-, -, \kappa/2, \kappa/2)$, with $\tau'' = (-; \kappa/2)$. The bundles are constructed as before — their existence requires the level k to be even. The orbits for this action are just those of Figure 1(c) except with an extra $\otimes Z$ everywhere. Thus the computation of $(-, -, \kappa/2, \kappa/2) K_*^{G \times Z \times Z}(G)$ mirrors that of (5.4) except for an extra $\otimes R_{\mathbb{Z}_2}$ everywhere: the full system $(-, -, \kappa/2, \kappa/2) K_0^{G \times Z \times Z}(G)$ consists of two copies of the nimrep $\mathbb{D}_{k/2+2}$, and $(-, -, \kappa/2, \kappa/2) K_1^{G \times Z \times Z}(G) = 0$. This is exactly what we would want.

To identify alpha-induction when $k/2$ is even, use the calculation of the Verlinde ring (5.3): comparing (5.1) and (5.4), we see that the natural map ${}^{\kappa} K_0^G(G) \rightarrow (-; \kappa/2) K_0^G(\bar{G})$ is just induction from $\text{SO}(2)$ to $\text{O}(2)$: $a^i \mapsto \kappa_i$. So this gives us $\alpha^{\pm} : \text{Ver}_k(G) \rightarrow (-; \kappa/2) K_0^G(\bar{G}) \otimes_{R_G} R_Z$. Let α^- be $(\text{Ind}_T^{\text{O}2}, +1)$, and α^+ be $(\text{Ind}_T^{\text{O}2}, \text{Res}_Z^T)$ (so these send a^i to $(\kappa_i, 1)$ and $(\kappa_i, (-1)^i)$ resp.). The neutral system is therefore the span of even i , as it should be.

In Table 4 we collect the data for the simple-current invariants for $\text{SU}(2)$. We also include, in the bottom two rows, the data for Fredenhagen's hypothetical models [29]. To form this table and provide the above story for $\text{SU}(2)$, we imported (rather than derived) information about the modular invariants. In the standard supersymmetric models associated to loop groups LG (e.g. those denoted 'CIZ' in Table 4), the fermions factor off, allowing everything to be describable by the bosons (i.e. the $\text{SU}(2)$ part). Fredenhagen proposed a model for $\text{SU}(2)$ where the centre $\pm I$ also acts nontrivially on fermions. In this case the modular invariant for $k/2$ odd is block-diagonal while that for $k/2$ even is an automorphism invariant, so we had to accommodate this in the table. As Fredenhagen noticed, his theory seemed to fit into the D-brane charge analysis of [12] if we assign to his theory the opposite H^1 -twist than that needed for the standard $\text{SO}(3)$ theory. To our knowledge it is still not yet clear whether Fredenhagen's model is a completely consistent supersymmetric theory but, at least from the point of view of K -homology, Fredenhagen's model seems to provide a coherent interpretation for the other H^1 -twist. In particular, his neutral system would recover the new ring structure on $(+; -, \kappa/2) K_{\bar{G}}^1(\bar{G}) \simeq (+; +, \kappa/2) K_0^{\bar{G}}(\bar{G})$ (for $\kappa/2$ even) mentioned in Remark 9.3 in [30]. One difference with the standard model is that $(+; \kappa/2) K_1^G(\bar{G}) \simeq -R_{\text{O}2}^1$ is 1-dimensional, not 0.

	nimrep	neutral system	full system
$k/2$ even : CIZ	$(-;\kappa/2) K_0^{\text{SU}2}(\text{SO}(3))$	$(-;+,\kappa/2) K_0^{\text{SO}3}(\text{SO}(3))$	$(-,-;\kappa/2,\kappa/2) K_0^{\text{SU}2 \times Z \times Z}(\text{SU}(2))$
$k/2$ odd : CIZ	$(-;\kappa/2) K_0^{\text{SU}2}(\text{SO}(3))$	$\kappa K_0^{\text{SU}2}(\text{SU}(2))$	$\kappa K_0^{\text{SU}2}(\text{SU}(2))$
$k/2$ even : Fr	$(+;\kappa/2) K_0^{\text{SU}2}(\text{SO}(3))$	$\kappa K_0^{\text{SU}2}(\text{SU}(2))$	$\kappa K_0^{\text{SU}2}(\text{SU}(2))$
$k/2$ odd : Fr	$(+;\kappa/2) K_0^{\text{SU}2}(\text{SO}(3))$	$(+;+,\kappa/2) K_0^{\text{SO}3}(\text{SO}(3))$	$(+,+;\kappa/2,\kappa/2) K_0^{\text{SU}2 \times Z \times Z}(\text{SU}(2))$

Table 4. Data for the $\text{SO}(3)$ theory, including Fredenhagen's hypothetical one

From this it is easy to guess what happens in general. Write $G_n = \text{SU}(n)$. Fix any divisor $d|n$ and level k , and write $n' = n/d$ and $\kappa = k+n$. The simple-current invariant $\mathcal{Z}_{\langle J^{n'} \rangle}$ exists iff $n'(n+1)k$ is even. Write Z_d for the order- d subgroup of the centre $Z(G_n) \simeq \mathbb{Z}_n$. The corresponding nimrep should be ${}^\tau K_0^{G_n}(G_n/Z_d) \simeq {}^\tau K^{G_n \times Z_d}(G_n)$ for some twist τ ; the neutral system should be ${}^{\tau'} K_0^{G_n/Z_{d'}}(G_n/Z_{d'})$ for some subgroup $Z_{d'}$ of Z_d and twist τ' ; the full system should be ${}^{\tau''} K_0^{\Delta_{G_n}^L \times \Delta_{G_n}^R}(G_n/Z_{d'} \times G_n/Z_{d'}) \simeq {}^{\tau''} K_0^{G_n^{\text{ad}} \times Z_{d'}^L \times Z_{d'}^R}(G_n) \simeq R_{Z_{d'}} \otimes_{\mathbb{Z}} {}^{\tau'''} K_0^{G_n^{\text{ad}} \times Z_{d'}^L}(G_n)$ for appropriate twists τ'', τ''' . These K -homology groups all vanish in degree 1. See Conjecture 3 below for more details.

A nimrep for $\mathcal{Z}_{\langle J^{n'} \rangle}$ has already been proposed in the conformal field theory literature, which we now describe. Write $[\lambda]_d$ for the orbit $\langle J^{n'} \rangle \lambda$, and $o_d(\lambda)$ for the largest positive integer o dividing d such that $J^{n/o} \lambda = \lambda$; hence $\text{stab}_{Z_d}(\lambda) = \langle J^{n/o_d(\lambda)} \rangle \simeq \mathbb{Z}_{o_d(\lambda)}$ and $\|[\lambda]_d\| = d/o_d(\lambda)$. For readability, we will usually drop the subscript on $[*]_d$ and o_d . Write $\mathcal{P}_d^k(G_n) = \{[\nu]_d : \nu \in P_+^k(G_n)\}$, the set of all $J^{n'}$ -orbits. Let $\mathcal{F}_{n,k,d}$ be the set of all $J^{n'}$ -fixed-points, i.e. all $\lambda \in P_+^k(G_n)$ such that $J^{n'} \lambda = \lambda$. Then $\mathcal{F}_{n,k,d}$ is nonempty iff d divides both n and k . The map $\mathcal{F}_{n,k,d} \rightarrow P_+^{k/d}(G_{n'})$ given by $\nu \mapsto \nu^{(d)} = (\nu_0; \nu_1, \dots, \nu_{n'-1})$ is a bijection.

As noted in [39], there are two possibilities which behave somewhat differently (recall $n'(n+1)k$ must be even):

Case A: *Either $n'(n+1)$ is even or the power of 2 dividing k exceeds that of n .* Then the diagonal entries of $\mathcal{Z}_{\langle J^{n'} \rangle}$ (i.e. the exponents of the corresponding nimrep) are precisely those $\mu \in P_+^k(G_n)$ with $Q_{J^{n'}}(\mu) \in \mathbb{Z}$, i.e. for which d divides $\sum_i i\mu_i$. Such an exponent μ has multiplicity $o_d(\mu)$.

Case B: *$n'(n+1)(k+1)$ is odd, and the power of 2 dividing n is at least as large as that dividing k .* Then the exponents can have $Q_{J^{n'}}(\mu) \in \frac{1}{2}\mathbb{Z}$ (see [39, Sect.1.2] for details).

Most of the study of nimreps for $\mathcal{Z}_{\langle J^{n'} \rangle}$ has been directed at the simpler and more common Case A, and in the following *we will restrict to Case A*. So for example this is automatically satisfied for $n = d = 3$ and $n = 4, d = 2$, but requires 4 to divide k for $n = d = 2$. Extending the following results to Case B, and then to the other G , should be a priority.

An obvious candidate for a nimrep compatible with $\mathcal{Z}_{\langle J^{n'} \rangle}$ would be the $J^{n'}$ -orbits $\text{Ver}_k(G_n)/\langle J^{n'} \rangle$, or equivalently the quotient of R_{G_n} by the ideal generated by the

fusion ideal $I_k(G_n)$ together with the terms $\rho_{\overline{J^{n'}\lambda}} - \rho_{\overline{\lambda}}$ for all $\lambda \in P_+^k(G_n)$. This $\text{Ver}_k(G_n)$ -module has a basis parametrised by the orbits $\mathcal{P}_d^k(G_n)$, with coefficients

$$\mathcal{N}_{\lambda, [\nu]}^{[\nu']} = \sum_{i=1}^{d/o(\nu)} N_{\lambda, \nu}^{J^{n'} i \nu'} = \frac{o(\nu)}{\gcd(o(\nu), o(\nu'))} \sum_{i=1}^{d/\text{lcm}(o(\nu), o(\nu'))} N_{J^{n'} i \lambda, \nu}^{\nu'}, \quad (5.7)$$

using (2.9). But in general this is not quite a nimrep: the asymmetry between ν, ν' means that the transpose condition of Definition 2 can fail for $\lambda = C\lambda$. Moreover, the dimension is wrong: although the exponents all have $Q_{J^{n'}}(\mu) \in \mathbb{Z}$ (which is what we want), they all have multiplicity 1. Indeed, $\text{Ver}_k(G_n)/\langle J^{n'} \rangle$ will be a nimrep compatible with $\mathcal{Z}_{\langle J^{n'} \rangle}$, iff there are no fixed-points, i.e. iff $\gcd(d, k) = 1$. Identifying the correct nimrep for $\mathcal{Z}_{\langle J^{n'} \rangle}$, when there are fixed-points, is more subtle: it requires *resolving* the fixed-points. The answer is finally provided in Theorem 4 below.

[4, 39] proposed a complicated formula for a nimrep $\mathcal{N}^{(n, k, d)}$ for $\mathcal{Z}_{\langle J^{n'} \rangle}$, using (2.11) with Ψ taken from an expression for the S -matrices of nonsimply-connected groups conjectured in [36]. The boundary states $a \in \mathcal{B}$ consist of pairs $([\nu]_d, l)$ where $[\nu]_d \in \mathcal{P}_d^k(G_n)$ and $l \in \mathbb{Z}_{o_d(\nu)}$. This component l resolves the fixed-point ν . In particular,

$$\Psi_{([\nu]_d, l), (\mu, i)} = \frac{\sqrt{d}}{o(\nu)\sqrt{o(\mu)}} \sum_{\delta | \gcd(o(\nu), o(\mu))} \xi_\delta S_{\nu^{(\delta)}, \mu^{(\delta)}}^{(\delta)} \sum_{\ell \in \mathbb{Z}_\delta^\times} e^{2\pi i \ell (l-i)/\delta}, \quad (5.8)$$

where $i \in \mathbb{Z}_{o(\mu)}$, $S^{(\delta)}$ is the S -matrix for $G_{n/\delta}$ at level k/δ , and ξ_δ is some root of unity depending on δ, n, k, d . By construction, the resulting $\mathcal{N}^{(n, k, d)}$ will be a nimrep compatible with $\mathcal{Z}_{\langle J^{n'} \rangle}$ iff the coefficients $\mathcal{N}_{\lambda, ([\nu]_d, l)}^{(n, k, d)}([\nu'], l')$ arising in (2.11) are all nonnegative integers. However, in this description both integrality and nonnegativity are highly unobvious. Using the *fixed-point factorisation* of [44], [40] found a relatively simple expression (given below) for some of these coefficients, and from this could prove integrality, but nonnegativity remained out of reach. In Theorem 4 below, we use this to find a simple global description for the $\text{Ver}_k(G_n)$ -module structure, making its relation to K -homology more evident, and allowing us to finally prove nonnegativity and establish the nimrep property.

We can make explicit the isomorphism $\text{stab}_{Z_d}(\nu) \simeq \mathbb{Z}_{o_d(\nu)}$ by writing ϕ_ν^l for the irrep sending $J^{hn/o(\nu)}$ to $e^{2\pi i l h/o(\nu)}$ for all $h \in \mathbb{Z}$ and each $l \in \mathbb{Z}_{o_d(\nu)}$; it is convenient to extend its definition to all of Z_d by defining $\phi_\nu^l(J^{hn'}) = 0$ whenever $J^{hn'} \notin \text{stab}_{Z_d}(\nu)$, i.e. whenever $d/o(\nu)$ does not divide h . Note that $\phi_\nu^l = \frac{o(\nu)}{d} \sum_{i=1}^{d/o(\nu)} \psi^{l+io(\nu)} \in \frac{o(\nu)}{d} R_{Z_d}$, where ϕ^l are the d irreps of Z_d . Identify the boundary state $([\nu]_d, l)$ with $([\nu]_d, \frac{d}{o(\nu)} \phi_\nu^l)$; the strange normalisation of ϕ_ν^l is needed for it to lie in R_{Z_d} , and will absorb the extra factors appearing in (5.7).

Theorem 4. *Fix $G_n = \text{SU}(n)$, level k , and divisor $d|n$. Write $\kappa = k + n$, $n' = n/d$, and Z_d for the order- d subgroup of the centre of G_n . Assume that Case A holds. Define $I_k^d(G_n)$ to be the ideal of $R_{G_n \times Z_d}$ generated by the fusion ideal $I_k(G_n)$, together with the terms $\rho_{J^{n'}\lambda} - \rho_\lambda$ for all $\lambda \in P_+^k(G_n)$, as well as $\rho_\lambda \otimes \phi - \rho_\lambda \otimes \phi'$, for any*

$\phi, \phi' \in R_{Z_d}$ which are equal when restricted to $\text{stab}_{Z_d}(\lambda)$. Then $R_{G_n \times Z_d}/I_k^d(G_n)$ is a $\text{Ver}_k(G_n)$ -module, where $\text{Ver}_k(G_n)$ acts by multiplication in the obvious way (i.e. ignoring the Z_d component). A basis for this is $([\nu]_d, l) = ([\nu]_d, \frac{d}{o_d(\nu)}\phi_\nu^l) \in \mathcal{B}$; in terms of this basis we have the coefficients

$$\mathcal{N}_{\lambda, ([\nu], l)}^{([\nu'], l')} = \sum_{j=1}^{d/\text{lcm}(o(\nu), o(\nu'))} \delta_{l, l'}^{\text{gcd}(o(\nu), o(\nu'))} N_{J^{n'} j \lambda, \nu}^{\nu'} \quad (5.9)$$

where $\delta_{l, l'}^m$ equals 1 if m divides $l - l'$, and 0 otherwise. This nimrep is equivalent to the one, $\mathcal{N}^{(n, k, d)}$, described above; in particular, $\mathcal{N}^{(n, k, d)}$ is a nimrep.

Proof. To prove this, first derive (5.9), which follows straightforwardly from (5.7). The expressions for the coefficients of $\mathcal{N}^{(n, k, d)}$ is much more difficult, but the Appendix of [40] uses the fixed-point factorisation of [44] to compute

$$\mathcal{N}_{\Lambda_m, ([\nu], l)}^{(n, k, d) ([\nu'], l')} = \sum_{j=1}^{d/\text{lcm}(o(\nu), o(\nu'))} \delta_{l, l'}^\Delta N_{J^{n'} j \Lambda_{m/\Delta}, \nu^{(\Delta)}}^{\nu^{(\Delta)}} \quad (5.10)$$

valid for all fundamental weights Λ_m and all $([\nu], l), ([\nu'], l') \in \mathcal{B}$, where $\Delta = \text{gcd}(o(\nu), o(\nu'))$ and $N^{(\Delta)}$ are the fusion coefficients for $\text{Ver}_{k/\Delta}(G_{n/\Delta})$. From the Pieri rule, or Claim (c) in [39, Sect.4.1], these fusions in $\text{Ver}_{k/\Delta}(G_{n/\Delta})$ coincide with the corresponding ones in $\text{Ver}_k(G_n)$, and we find that $\mathcal{N}_{\Lambda_m, ([\nu], l)}^{(n, k, d) ([\nu'], l')} = \mathcal{N}_{\Lambda_m, ([\nu], l)}^{([\nu'], l')}$ for all m, ν, ν', l, l' .

Since these two $\text{Ver}_k(G_n)$ -modules have identical formulas for multiplication by the fundamental weights Λ_m , which generate $\text{Ver}_k(G_n)$, they are isomorphic as $\text{Ver}_k(G_n)$ -modules, and in fact $\mathcal{N}_\lambda = \mathcal{N}_\lambda^{(n, k, d)}$ for all $\lambda \in P_+^k(G_n)$. Therefore the coefficients of $\mathcal{N}^{(n, k, d)}$ are nonnegative integers, and so $\mathcal{N} = \mathcal{N}^{(n, k, d)}$ is indeed a nimrep compatible with $\mathcal{Z}_{\langle J^{n'} \rangle}$. QED

Theorem 4 provides a massive simplification of the original expressions for $\mathcal{N}^{(n, k, d)}$ in [4, 39]. Although integrality of the matrix entries of $\mathcal{N}^{(n, k, d)}$ was established in [40], nonnegativity was out-of-reach. We were led to Theorem 4 by trying to match the conjectured $\mathcal{N}^{(n, k, d)}$ to the K -homology ${}^\tau K_*^{G \times Z_d}(G)$. Working out this nimrep in Case B is a natural task. The fixed-point factorisation of [44] continues to hold, so one could extend the nimrep coefficient calculations of [40, App.B] to Case B, although this would be technically challenging, and then try to reinterpret the result globally as a quotient of a representation ring. Note though that Theorem 4 will fail as stated for Case B: in particular, not all exponents of $\mathcal{Z}_{\langle J^{n'} \rangle}$ will now obey $Q_{J^{n'}}(\mu) \in \mathbb{Z}$, so the simple-current $J^{n'} \in \text{Ver}_k(G_n)$ will not act trivially.

The charge-group for $\mathcal{N}^{(n, k, d)}$, in Case A, was conjectured in [40] to be

$$\mathbb{Z}_M \oplus \bigoplus_{p|\text{gcd}(d, M)} \bigoplus_{i=1}^{\delta} (p^i - p^{i-1}) \cdot \mathbb{Z}_{p^{\min(\mu, \nu-i+1)}} \quad (5.11)$$

where M is the gcd of the dimensions of all weights in the fusion ideal, and $p^\nu \parallel n$, $p^\delta \parallel d$ and $p^\mu \parallel M$. The first sum runs over all primes dividing both d and M . The charge assignments which should generate this, and the evidence supporting this conjecture, is discussed in [40, Sect.4.2]. With the new global interpretation of $\mathcal{N}^{(n,k,d)}$ coming from Theorem 4, it is very possible this conjecture can now be proved. The charges and charge-groups in Case 2 are discussed in [39, Sect.5].

Before we can explicitly introduce the relevant K -homology groups, we need to understand the appropriate cohomology groups. Note from (2.3) that $H_{G_n}^1(G_n/Z_d; \mathbb{Z}_2)$ is \mathbb{Z}_2 or 0 depending on whether or not d is even, and $H_{G_n}^3(G_n/Z_d; \mathbb{Z}) \simeq \mathbb{Z}$. The usual spectral sequence argument gives $H_{G_n/Z_d}^3(G_n/Z_d; \mathbb{Z}) \simeq \mathbb{Z}_d \times \mathbb{Z}$, using $H_{G_n/Z_d}^*(pt; \mathbb{Z}) \simeq \mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z}_d \oplus \mathbb{Z} \oplus \dots$ and $H^3(G_n/Z_d; \mathbb{Z}) \simeq \mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z} \oplus \dots$; likewise $H_{G_n/Z_d}^1(G_n/Z_d; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ or 0, again depending on whether or not d is even. Finally, $H_{G_n \times Z_d \times Z_d}^3(G_n; \mathbb{Z}) \simeq R_{Z_d} \otimes_{\mathbb{Z}} H_{G_n}^3(G_n/Z_d; \mathbb{Z})$, and $H_{G_n \times Z_d \times Z_d}^1(G_n; \mathbb{Z}_2) \simeq \mathbb{Z}_2(Z_d)$ or 0, depending again on whether or not d is even.

Conjecture 3. *Use the same notation as Theorem 4; we require $n'(n+1)k$ is even (necessary for the existence of $\mathcal{Z}_{\langle Jn' \rangle}$) but don't assume Case A. Write $t = 1$ or 2 when $n'(n+1)$ is even respectively odd. Fix $s \in \mathbb{Z}_2$ (which can be dropped if d is odd).*

(a) $Z(G_n) \simeq \mathbb{Z}_n$ acts on the G_n on G_n bundle, and the corresponding action on ${}^\kappa K_0^{G_n}(G_n) \simeq \text{Ver}_k(G_n)$ is by simple-currents.

(b) ${}^{(s;\kappa/t)}K_1^{G_n}(G_n/Z_d) = 0$ and ${}^{(s;\kappa/t)}K_0^{G_n}(G_n/Z_d)$ is a nimrep for the simple-current modular invariant $\mathcal{Z}_{\langle Jn' \rangle}$. This should agree with the (conjectured) nimrep $\mathcal{N}^{(n,k,d)}$.

(c) Let $d_0 = \gcd(d, kn')$ if d is odd, and $d_0 = \gcd(d, kn'/2)$ if d is even. Write Z_0 for the order- d_0 subgroup of the centre of G_n . Then the neutral system is ${}^{(s;\tau',\kappa/d_0)}K_0^{G_n/Z_0}(G_n/Z_0)$ for some choice of $\tau' \in \mathbb{Z}_{d_0}$ (s can be dropped if d_0 is odd).

The full system is ${}^{(s,\dots,s;\kappa/t,\dots,\kappa/t)}K_0^{G_n \times Z_0^L \times Z_0^R}(G_n) \simeq R_{Z_0} \otimes_{\mathbb{Z}} {}^{(s;\kappa/t)}K_0^{G_n}(G_n/Z_0)$, i.e. d_0 copies of ${}^{(s;\kappa/t)}K_0^{G_n}(G_n/Z_0)$ (again, s can be dropped if d_0 is odd). The corresponding twisted equivariant K_1 -homology groups all vanish. Write β for the map ${}^\kappa K_0^{G_n}(G_n) \rightarrow {}^{(s;\kappa/t)}K_0^{G_n}(G_n/Z_0)$ coming from the projection $G_n \rightarrow G_n/Z_0$. Then $\alpha^+ : \text{Ver}_k(G_n) \rightarrow R_{Z_0} \otimes_{\mathbb{Z}} {}^{(s;\kappa/t)}K_0^{G_n}(G_n/Z_0)$ can be chosen to be $[\lambda] \mapsto (1, \beta(\lambda))$ while α^- is $(\overline{\text{Res}}_{Z_0}^{G_n}, \beta)$ ($\overline{\text{Res}}_{Z_0}^{G_n}$ was defined earlier this section). This map β can also be interpreted as ${}^\kappa K_0^{G_n}(G_n) \rightarrow {}^{(s;\kappa)}K_0^{G_n \times Z_0}(G_n)$ where Z_0 acts trivially by adjoint; in K -theory alpha-induction would involve induction from G_n to $G_n \times Z_0$.

(d) The D -brane charge group here is ${}^{\kappa/t}K_0(G_n/Z_d)$. The assignment of charges is given by the forgetful map ${}^{(s;\kappa/t)}K_0^{G_n}(G_n/Z_d) \rightarrow {}^{\kappa/t}K_0(G_n/Z_d)$.

The possibility of choosing either sign for s is suggested by the Fredenhagen model [29].

Something similar to Conjecture 3 will hold for all other compact connected groups, and this should be worked out. As a first step, the analogue of fixed-point factorisation has been found for all G [3]. That there can possibly be subtleties is hinted by $G = \text{Spin}(8)$ at level 2 (so $\kappa = 8$), for Z the full centre $\mathbb{Z}_2 \times \mathbb{Z}_2$. The

corresponding modular invariant is

$$\mathcal{Z} = |\chi_{(2;0000)} + \chi_{(0;2000)} + \chi_{(0;0020)} + \chi_{(0;0002)}|^2 + 4|\chi_{(0;0100)}|^2, \quad (5.12)$$

using obvious notation. The neutral system is $^{17}K_0^{E_7}(E_7)$, which is 2-dimensional (and not 5-dimensional), with sigma-restriction sending those 2 primaries to $\chi_{(2;0000)} + \chi_{(0;2000)} + \chi_{(0;0020)} + \chi_{(0;0002)}$ and $2\chi_{(0;0100)}$ (and not the naive guess $\chi_{(2;0000)} + \chi_{(0;2000)} + \chi_{(0;0020)} + \chi_{(0;0002)}$, $\chi_{(0;0100)}$, $\chi_{(0;0100)}$, $\chi_{(0;0100)}$, $\chi_{(0;0100)}$). The proper way to think of this is described at the end of [36, Sect.6.4]. It would be interesting (and not too difficult) to determine whether ${}^\tau K_0^{\text{PSpin}(8)}(\text{PSpin}(8))$ for the appropriate twist τ recovers those 2 dimensions (and not for instance the 5).

5.3 Mixing outer automorphisms and simple-currents

The final generic source of modular invariants results from combining simple current invariants and outer automorphisms. Surprisingly, the analysis here is much simpler than in section 5.2. In this subsection we will restrict to the modular invariant $\mathcal{Z}_{\langle J^{n'} \rangle}^c = C\mathcal{Z}_{\langle J^{n'} \rangle} = \mathcal{Z}_{\langle J^{n'} \rangle}C$ for $G_n = \text{SU}(n)$, where C is charge-conjugation and $n' = n/d$ for some $d|n$. We retain all notation from section 5.2. In particular, recall the $\text{Ver}_k(G_n)$ -nimrep R_{Z_d} , with multiplication $\lambda.\phi_d^l = \phi_d^{l+\sum_j j\lambda_j}$. We'll begin by describing what appears in the CFT literature.

Consider first d odd. Let \mathcal{N}^c be any nimrep for $\mathcal{Z} = I_c$. It was proved in [40, Sect.2.1] that $R_{Z_d} \otimes_{R_{G_n}} \mathcal{N}^c$ is a nimrep compatible with $\mathcal{Z}_{\langle J^{n'} \rangle}^c$. In particular, recall the conjectured nimrep \mathcal{N}^{GG} of section 4.2. Then $R_{Z_d} \otimes \mathcal{N}^{GG}$ will be a nimrep for $\mathcal{Z}_{\langle J^{n'} \rangle}^c$ iff the coefficients $\mathcal{N}_{\lambda,x}^{GG,y}$ are all nonnegative, as was conjectured. Denoting the boundary states of $\mathcal{N} = R_{Z_d} \otimes \mathcal{N}^{GG}$ by $(x, l) = (x, \phi_d^l)$, for $x \in P_+^k(A_{n-1}^{(2)})$ and $l \in \mathbb{Z}_d$, we obtain the coefficients $\mathcal{N}_{\lambda,(x,l)}^{(y,l')} = \mathcal{N}_{\lambda,x}^{GG,y} \delta_{l',l+\sum_j j\lambda_j}^d$ for δ^d defined in Theorem 4. The charge-group was proved in [40, Sect.2.2.2] to be (5.11), subject only to the validity of Conjecture 1(a) for $G = \text{Spin}(2n+1)$, and the charge assignments were all identified.

When d is even, this construction fails because $J^{n/2}$ is an exponent of I_c , so the vacuum 1 has multiplicity 2, not 1, in $R_{Z_d} \otimes \mathcal{N}^c$. But this also suggests the cure, at least when n' is even. The $\text{Ver}_k(G_n)$ -module $R_{Z_{2d}} \otimes \mathcal{N}^c$ consists of two identical copies of what we want. Restrict for example to the submodule spanned (over \mathbb{Z}) by the boundary states (x, ϕ^l) , $x \in P_+^k(A_{n-1}^{(2)})$, where $l \equiv \sum i x_i \pmod{2}$. Assuming again that the coefficients of \mathcal{N}^{GG} are nonnegative for G_n , this submodule of $R_{Z_{2d}} \otimes \mathcal{N}^{GG}$ will be a nimrep for $\mathcal{Z}_{\langle J^{n'} \rangle}^c$. Its charge-group was conjectured in [40] to be (5.11) — see section 3.2 there for some partial results and supporting evidence.

When $n'(n+1)$ is odd, the situation is more complicated because the exponents of $\mathcal{Z}_{\langle J^{n'} \rangle}^c$ are not directly related to those of I_c (by contrast, the exponents for $\mathcal{Z}_{\langle J^{n'} \rangle}^c$, when d is odd say, are $J^{jn'}\mu$ of multiplicity $o(\mu)$ for any $\mu = C\mu$). This means there won't be a direct relation between the nimreps for I_c and $\mathcal{Z}_{\langle J^{n'} \rangle}^c$. Indeed,

when $n'(n+1)$ is odd, we will no longer in general have agreement between the charge-groups of $\mathcal{Z}_{\langle J^{n'} \rangle}$ and $\mathcal{Z}_{\langle J^{n'} \rangle}^c$: e.g. as mentioned in [40], for $n = k = d = 4$, $\mathcal{M}(\mathcal{N}^{(4,4,4)}) \simeq \mathbb{Z}_4 \times \mathbb{Z}_2^2$ while $\mathcal{M}(\mathcal{N}^{(4,4,4)^c}) \simeq \mathbb{Z}_4^2$.

Now let's turn to a K -homological interpretation, focussing on the simpler case where d is odd. The groups $H_{G_n^{\text{ad}_c}}^3(G_n/Z_d; \mathbb{Z}) \simeq H_{G_n^{\text{ad}_c} \times Z_d^L}^3(G_n; \mathbb{Z}) \simeq \mathbb{Z}$ and $H_{G_n^{\text{ad}_c}}^1(G_n/Z_d; \mathbb{Z}_2) \simeq H_{G_n^{\text{ad}_c} \times Z_d^L}^1(G_n; \mathbb{Z}_2) \simeq \mathbb{Z}_{\gcd(d,2)}$, calculated as before. The Dixmier-Douady bundles should be constructable by a combination of the methods of sections 4.2 and 5.2.

Theorem 5. *Let $G_n = SU(n)$, $d|n$, $n' = n/d$, Z_d be the order- d central subgroup, and $\kappa = k+n$ as before. Assume d to be odd. Then ${}^\kappa K_{G_n^{\text{ad}_c}}^i(G_n/Z_d)$ is naturally isomorphic to $R_{Z_d} \otimes_{R_{G_n}} {}^\kappa K_{G_n^{\text{ad}_c}}^i(G_n)$ and vanishes for $i = (n+1)(n-2)/2$. ${}^\kappa K_{G_n^{\text{ad}_c}}^*(G_n/Z_d)$ will be a nimrep compatible with $\mathcal{Z}_{\langle J^{n'} \rangle}^c$, provided Conjecture 2(a) holds for $G = G_n$.*

Proof. First note that, for any subgroup Z' of $Z(G_n)$, ${}^\kappa K_{(G_n \times Z')^{\text{ad}_c}}^*(G_n) \simeq R_{Z'} \otimes_{R_{G_n}} {}^\kappa K_{G_n^{\text{ad}_c}}^*(G_n)$ as R_{G_n} -modules, using (2.6) and the automorphism of $G_n \times Z'$ sending $(g, z') \mapsto (gz', z')$. R_{G_n} multiplies $R_{Z'}$ through restriction, i.e. through $\overline{\text{Res}}_{Z'}^{G_n}$.

Since d is odd, we can write $\sqrt{z} := z^{(d+1)/2}$ for any $z \in Z_d$. The key observation is that $(g, z).h = \sqrt{z}gh(c\sqrt{z}g)^{-1}$ since complex conjugation c sends \sqrt{z} to $1/\sqrt{z}$. Hence ${}^\kappa K_{G_n^{\text{ad}_c} \times Z_d^L}^*(G_n) \simeq {}^\kappa K_{(G_n \times Z_d)^{\text{ad}_c}}^*(G_n) \simeq R_{Z_d} \otimes {}^\kappa K_{G_n^{\text{ad}_c}}^*(G_n)$ as desired.

The vanishing in degree $i = (n+1)(n-2)/2$ follows from Theorem 3. The nimrep statement follows from [40, Sect.2.1]. QED

The full system, neutral system and sigma-induction is independent of the twist ω , and so are as in section 5.2. Alpha-induction here is obtained by combining sections 4.2 and 5.2: $\alpha_\lambda^+ = (1, \beta(\lambda)) \in R_{Z_0} \otimes {}^\kappa K_0^{G_n}(G_n/Z_0)$ and $\alpha_\lambda^- = (\overline{\text{Res}}_{Z_0}^{G_n}, \beta \circ c)$ where β and Z_0 are as in Conjecture 3. The obvious guess for the assignment of D-brane charges is ${}^\tau K_{G_n}^{n(n-1)/2}(G_n/Z_d) \rightarrow {}^\tau K^{n(n-1)/2}(G_n/Z_d)$.

There should be a similar story when both d and n' are even. Then $2d$ will also divide n , so Z_{2d} also exists. Of course Z_{2d} must be cyclic; fix a generator z . Then $(g, z^{2j}).h = (z^j g)hc(z^j g)^{-1}$ as before so we find ${}^{(0;\kappa)} K_{G_n^{\text{ad}_c}}^*(G_n/Z_d) \simeq {}^{(0;\kappa)} K_{((G_n \times Z_{2d})/1 \times Z_2)^{\text{ad}_c}}^*(G_n)$. On the other hand, ${}^{(0;\kappa)} K_{(G_n \times Z_{2d})^{\text{ad}_c}}^*(G_n)$ will be isomorphic to $R_{Z_{2d}} \otimes_{R_{G_n}} {}^\kappa K_{G_n^{\text{ad}_c}}^*(G_n)$, and as a R_{G_n} -module has a \widehat{Z}_2 -grading obtained by restricting the equivariance to Z_2 , which always acts trivially. This is the K -theoretic explanation for the aforementioned reducibility of $R_{Z_{2d}} \otimes_{R_{G_n}} \mathcal{N}^c$. We'd expect ${}^{(0;\kappa)} K_{G_n^{\text{ad}_c}}^*(G_n/Z_d)$ to be isomorphic to half of $R_{Z_{2d}} \otimes_{R_{G_n}} {}^\kappa K_{G_n^{\text{ad}_c}}^*(G_n)$, and thus to describe K -theoretically the nimrep of [40].

6 Exceptional modular invariants

6.1 Conformal embeddings

Section 2.3 reminds us that $G = \mathrm{SU}(2)$ has 3 exceptional modular invariants. Two of these are due to *conformal embeddings* (defined in section 2.3), so it is to these we first turn. Conformal embeddings for the finite groups were described in [22].

In [22] we proposed that conformal embeddings $H_k \rightarrow G_1$ could be related to K -homology ${}^\tau K_*^{H^{\mathrm{ad}}}(G)$. We studied in detail e.g. the conformal embeddings $\mathrm{SU}(2)_4 \rightarrow \mathrm{SU}(3)_1$ and $\mathrm{SU}(2)_{10} \rightarrow \mathrm{Sp}(4)_1$, which give rise to the modular invariants called \mathbb{D}_4 and \mathbb{E}_6 , respectively, in the $\mathrm{SU}(2)_k$ list of Cappelli *et al* [14]. Perhaps the most interesting observation to come out of this analysis was that the largest finite stabiliser in this adjoint action of $\mathrm{SU}(2)$ on $\mathrm{SU}(3)$ resp. $\mathrm{Sp}(4)$, is called \mathbb{D}_4 resp. \mathbb{E}_6 on McKay's list [51]. In this subsection we propose an alternative K -homology, namely ${}^\tau K_*^{\Delta_H^L \times \Delta_H^R}(G \times G)$ with the diagonal action $(h_L, h_R)(g_1, g_2) = (h_L g_1 h_R^{-1}, h_L g_2 h_R^{-1})$. The previous observation about finite stabilisers would persist in this new picture: if $K < H$ is the stabiliser of $g \in G$, then the isomorphic copy $\{(k, k)\} < H \times H$ stabilizes $(g, z) \in G \times G$ for any $z \in Z(G)$.

For convenience we restrict in the following to examples calculable in K -theory through the Hodgkin spectral sequence (recall section 2.2).

Consider first one of the simplest possible conformal embeddings: $T_2 \subset \mathrm{SU}(2)_1$. Here $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $H = \mathrm{SO}(2) \times \mathrm{SO}(2)$. Write $R_G = \mathbb{Z}[\sigma, \tau]$, $R_H = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$ using obvious notation. The Verlinde ring $\mathrm{Ver}_1(\mathrm{SU}2)$ is $R_G/(\sigma^2 - 1, \sigma - \tau)$. The restriction map is $\sigma \mapsto a + a^{-1}$, $\tau \mapsto b + b^{-1}$. The free resolution is

$$0 \rightarrow R_G \xrightarrow{g} R_G^2 \xrightarrow{f} R_G \rightarrow R_G/(\sigma^2 - 1, \sigma - \tau) \rightarrow 0, \quad (6.1)$$

where $f(a_1, a_2) = (\sigma^2 - 1)a_1 + (\sigma - \tau)a_2$ and $g(c) = (-\sigma + \tau, \sigma^2 - 1)c$. The straightforward calculation shows that all $E_{p,q}^2 = 0$ except $E_{0,\mathrm{odd}}^2 = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]/(a^2 + 1 + a^{-2}, a + a^{-1} - b - b^{-1})$, which is 8-dimensional, with coset representatives $1, a, b, a^2, ab, b^2, a^3, a^2b$. Thus ${}^\tau K_{\Delta_T^L \times \Delta_T^R}^*(G \times G) \simeq 0 \oplus \mathbb{Z}^8$.

This result can also be obtained more elegantly from the ‘maximal rank’ argument of section 3.3 of [22]. From this we see ${}^\tau K_{T \times T}^0(G \times G)$ decomposes naturally into 4 copies of the full system $\mathrm{Ver}_2(T)$, where ‘4’ is the order of the Weyl group of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, which on $R_{T \times T}$ acts by $a \mapsto a^{s_1}$, $b \mapsto b^{s_2}$ for some choice of signs s_1, s_2 .

By comparison, the K -group ${}^\tau K_{T^{\mathrm{ad}}}^*(\mathrm{SU}2)$ was calculated in sections 3.2 and 3.3 of [22], and found to give 2 copies of the full system, where ‘2’ is the order of the Weyl group of $\mathrm{SU}(2)$.

Thus it would appear that modelling this conformal embedding by the diagonal action rather than the adjoint action gains little, and in fact makes things a little worse. However, consider instead the $E_{8,1} \rightarrow \mathrm{SU}(9)_1$ conformal embedding. Then by identical arguments ${}^\tau K_{\mathrm{SU}9^{\mathrm{ad}}}^*(E_8) \simeq 0 \oplus \mathbb{Z}^{1920}$ while ${}^\tau K_{\Delta_{\mathrm{SU}9}^L \times \Delta_{\mathrm{SU}9}^R}^*(E_8 \times E_8) \simeq 0 \oplus$

$\mathbb{Z}^{3686400}$. The corresponding modular invariant is

$$\mathcal{Z} = |\chi_{000000000} + \chi_{001000000} + \chi_{000001000}|^2$$

so the full system is 9-dimensional. Thus ${}^\tau K_{\text{SU}9}^1(E_8)$ cannot consist of a number of copies of the full system, whereas ${}^\tau K_{\text{SU}9 \times \text{SU}9}^1(E_8 \times E_8)$ very possibly could.

In [22] we speculated that conformal embeddings would be realised K -homologically using the adjoint action, by ${}^\tau K_*^H(S)$ for some H -invariant submanifold S_0 of G , typically smaller than G . But at least sometimes the diagonal action could permit a much more direct interpretation: its K -homology more often can consist of a certain number of copies of the full system.

Consider now the \mathbb{E}_6 modular invariant in the A-D-E list of [14]: this comes from the conformal embedding $\text{SU}(2)_{10} \rightarrow \text{Sp}(4)_1$. The full system is 12-dimensional, consisting of two copies of the \mathbb{E}_6 diagram. In [22] we considered ${}^\tau K_{\text{SU}2}^{\text{SU}2^{\text{ad}}}(\text{Sp}4)$ in detail, determining it was $2 + 2$ dimensional. We would have preferred two copies of the \mathbb{E}_6 diagram.

But consider instead the subgroup $H = \text{SU}(2) \times \text{SU}(2)$ of $G = \text{Sp}(4) \times \text{Sp}(4)$, where the embedding $\text{SU}(2) \subset \text{Sp}(4)$ is the 4-dimensional irreducible $\text{SU}(2)$ -representation given explicitly in [22]. The basic representation rings are $R_{\text{SU}2} = \mathbb{Z}[\sigma]$ and $R_{\text{Sp}4} = \mathbb{Z}[s, v]$ where $\sigma = \sigma_2$, and s, v are the spinor and vector representations respectively. We write $R_H = \mathbb{Z}[\sigma, \sigma']$ and $R_G = \mathbb{Z}[s, v, s', v']$ using obvious notation. We will use the Hodgkin spectral sequence to compute ${}^\tau K_{\Delta_{\text{SU}2}^L \times \Delta_{\text{SU}2}^R}^*(\text{Sp}4 \times \text{Sp}4)$. R_G multiplies R_H using the restrictions $s \mapsto \sigma_4 = \sigma^3 - 2\sigma$, $v \mapsto \sigma_5 = \sigma^4 - 3\sigma^2 + 1$. As an R_G -module, we have

$${}^\tau K_G^0(X) = \text{Ver}_1(\text{Sp}4) = R_G/(w, x, y, z) \quad (6.2)$$

where for later convenience we write $w = s^2 - v - 1$, $x = sv - s$, $y = v - v'$, $z = s - s'$. Note that these restrict to $(\sigma^2 - 2)(\sigma^4 - 3\sigma^2 + 1)$, $(\sigma^2 - 2)\sigma^3(\sigma^2 - 3)$, $(\sigma^2 - \sigma'^2)(\sigma^2 + \sigma'^2 - 3)$, $(\sigma - \sigma')(\sigma^2 + \sigma\sigma' + \sigma'^2 - 2)$ respectively. A free resolution of $\text{Ver}_1(\text{Sp}4)$ is

$$0 \rightarrow R_G \xrightarrow{i} R_G^4 \xrightarrow{h} R_G^6 \xrightarrow{g} R_G^4 \xrightarrow{f} R_G \rightarrow R_G/(w, x, y, z) \rightarrow 0, \quad (6.3)$$

where $f(a_1, a_2, a_3, a_4) = a_1 w + a_2 x + a_3 y + a_4 z$, $i(d) = (z, -y, -x, w)d$, and

$$\begin{aligned} g(b_1, \dots, b_6) = & (x, -w, 0, 0)b_1 + (-y, 0, w, 0)b_2 + (z, 0, 0, -w)b_3 \\ & + (0, -y, x, 0)b_4 + (0, z, 0, -x)b_5 + (0, 0, -z, y)b_6, \end{aligned} \quad (6.4)$$

$$\begin{aligned} h(c_1, c_2, c_3, c_4) = & (y, x, 0, -w, 0, 0)c_1 + (z, 0, -x, 0, w, 0)c_2 + (0, z, y, 0, 0, w)c_3 \\ & + (0, 0, 0, z, y, x)c_4. \end{aligned} \quad (6.5)$$

By definition, $E_{p,0}^2 = \text{Tor}_p^{R_G}(R_H, R_G/(w, x, y, z))$ and $E_{p,1}^2 = 0$ for all $p \geq 0$. We obtain $E_{0,0}^2 = \mathbb{Z}[\sigma, \sigma']/(\sigma^2 - 2, \sigma'^2 - 2) =: Q$,

$$\begin{aligned} E_{1,0}^2 = & \text{span}_{\mathbb{Z}}\{(\sigma^5 - 3\sigma^3, -\sigma^4 + 3\sigma^2 - 1, 0, 0), ((\sigma\sigma' - \sigma^2)(\sigma\sigma' + 1), (\sigma - \sigma')(\sigma\sigma' + 1), \sigma', 1 - \sigma'^2)\} Q, \\ E_{2,0}^2 = & (\sigma\sigma'^2 - \sigma^2\sigma' + \sigma' - \sigma, 3\sigma^3\sigma' - \sigma^5\sigma', (3\sigma^3 - \sigma^5)(\sigma'^2 - 1), \sigma^4\sigma' - 3\sigma^2\sigma' + \sigma', (\sigma^4 - 3\sigma^2 + 1)(\sigma'^2 - 1), 0) Q, \end{aligned}$$

and $E_{p,0}^2 = 0$ for $p \geq 3$. This sequence stabilises already at $E_{p,q}^2$, and we find that both groups ${}^\tau K_{\Delta_{\text{SU}2}^L \times \Delta_{\text{SU}2}^R}^i(\text{Sp}(4) \times \text{Sp}(4))$ are 8-dimensional, and as an $R_{\text{SU}2 \times \text{SU}2}$ -module both are isomorphic to two copies of $\mathbb{Z}[\sigma, \sigma']/(\sigma^2 - 2, \sigma'^2 - 2)$.

In the examples of sections 3, 4 and 5, the action of the Verlinde ring on the nimrep and full system comes from the push-forward of the inclusion of the identity of the group, i.e. from the natural action of the representation ring on the equivariant K -groups. Here, this R_H -action does not factor through to $\text{Ver}_{10}(\text{SU}2) \simeq R_H/(\sigma^{11} - 10\sigma^9 + 36\sigma^7 - 56\sigma^5 + 35\sigma^3 - 6\sigma, \sigma - \sigma')$ (the problem is the $\sigma - \sigma'$). This module structure should come from the product discussed in section 2.2.

In [22] we modeled the conformal embedding $\text{SU}(2)_4 \rightarrow \text{SU}(3)_1$ by ${}^\tau K_{\text{SU}2^{\text{ad}}}^*(\text{SU}3)$ and got $1 + 1$ dimensions (the full system is 8-dimensional, consisting of two copies of the \mathbb{D}_4 graph). But we now see that a better approximation should be ${}^\tau K_{\Delta_{\text{SU}2}^L \times \Delta_{\text{SU}2}^R}^*(\text{SU}3 \times \text{SU}3)$. It would take some work to compute this, since the Hodgkin spectral sequence doesn't apply here (this embedding is of $\text{SO}(3)$, not $\text{SU}(2)$). Let us make some qualitative comments. The dimension of these K -groups should surely be greater than $1 + 1$, as the above examples indicate. To what extent can we hope to see \mathbb{D}_4 's in these K -homological groups? The distinguishing feature of the (unextended) \mathbb{D}_4 diagram is the S_3 symmetry of the three endpoints, which fixes the central vertex. This S_3 symmetry appears naturally here: on each factor space $\text{SU}(3)$ it is generated by multiplying by a scalar matrix $\omega^i I$ (these form the centre of $\text{SU}(3)$), and by complex conjugation — all of these commute with the $\text{SU}(2)$ action. So it is not impossible to imagine \mathbb{D}_4 lurking here.

In hindsight it is clear why we are not getting the correct answer spot-on here: we are ignoring the spinors. More precisely, the dimension shift $\dim(G)$ in the degree, and the shift by h^\vee of the level, in the identification $\text{Ver}_k(G) \simeq {}^{k+h^\vee} K_{G^{\text{ad}}}^{\dim G}(G)$ (for G simply-connected, connected and compact) come from the implicit presence of a $\text{Cliff}(L\mathfrak{g}^*)$ -module, which comes along for the ride. However it appears it cannot be factored off so simply in the conformal embedding context: in particular the level shift does not occur, as was demonstrated in the level calculations in [22, Sect.2.3]. Considering for concreteness the $\text{SU}(2)_4 \rightarrow \text{SU}(3)_1$ conformal embedding, we know from [22, Sect.2.3] that the twist on the $\text{SU}(2)$ and $\text{SU}(3)$ parts really should be 4 and 1 respectively, and not $4 + 2$ and $1 + 3$. So this means the spinors cannot be so easily ignored. Including them will account for at least some of the discrepancy found above.

6.2 The \mathbb{E}_7 modular invariant of $\text{SU}(2)$

The preceding sections address all modular invariants of $G = \text{SU}(2)$, except for \mathbb{E}_7 at level 16. Its direct interpretation would be as a twist of the even part of \mathbb{D}_{10} ; in this sense it is exceptional in that no other \mathbb{D}_n have such a twist. But in another sense the \mathbb{E}_7 modular invariant is not really exceptional: it belongs to an infinite sequence at $\text{SO}(n)$ level 8 coming from a combination of *level-rank duality* (which relates $\text{SO}(n)_k$ with $\text{SO}(k)_n$, at least when kn is even) and $\text{SO}(8)$ triality. In this subsection we

obtain a partial K -theoretic realisation of \mathbb{E}_7 by approaching it in this way.

The neutral system (maximal chiral extension) for \mathbb{E}_7 is \mathbb{D}_{10} , i.e. ${}^{8+1}K_{\text{SO}_3}^1(\text{SO}(3))$; the branching rules (sigma-restriction) ${}^{8+1}K_{\text{SO}_3}^1(\text{SO}(3)) \rightarrow {}^{16+2}K_{\text{SU}_2}^1(\text{SU}(2))$ is $\bar{0} \leftrightarrow 0 + 16$, $\bar{1} \leftrightarrow 2 + 14$, $\bar{2} \leftrightarrow 4 + 12$, $\bar{3} \leftrightarrow 6 + 10$, and the resolved fixed-point $\bar{4}, \bar{4}' \leftrightarrow 8$. Level-rank duality comes from the conformal embedding $\text{SO}(24)_1 \leftarrow \text{SO}(8)_3 \oplus \text{SO}(3)_8$, which has branching rules [66]

$$0 \mapsto (0000)(0 + 16) + (2000)(2 + 14) + (0100)(4 + 12) + (1011)(6 + 10) + ((0020) + (0002))8 \quad (6.6)$$

$$v \mapsto (3000)(0 + 16) + (1000)(2 + 14) + (1100)(4 + 12) + (0011)(6 + 10) + ((1020) + (1002))8 \quad (6.7)$$

$$s \mapsto (0003)(0 + 16) + (0021)(2 + 14) + (0101)(4 + 12) + (1010)(6 + 10) + ((2001) + (0001))8 \quad (6.8)$$

$$c \mapsto (0030)(0 + 16) + (0012)(2 + 14) + (0110)(4 + 12) + (1001)(6 + 10) + ((2010) + (0010))8 \quad (6.9)$$

using obvious notation. Thus we get the identification $[(0000)] \leftrightarrow \bar{0}$, $[(2000)] \leftrightarrow \bar{1}$, $[(0100)] \leftrightarrow \bar{2}$, $[(1011)] \leftrightarrow \bar{3}$, $[(0020)] \leftrightarrow \bar{4}$, $[(0002)] \leftrightarrow \bar{4}'$ between the K -groups ${}^{3+9}K_{\text{PSpin}_8}^0(\text{PSpin}(8))$ and ${}^{8+1}K_{\text{SO}_3}^1(\text{SO}(3))$, equivalently between the corresponding orbits of simple-currents. The \mathbb{E}_7 modular invariant is recovered from the diagonal modular invariant of $\text{SO}(24)_1$ and the triality modular invariant of $\text{SO}(8)_3$, by contraction [16].

The full system of \mathbb{E}_7 was obtained in [56, 10]; we reproduce the figure below:

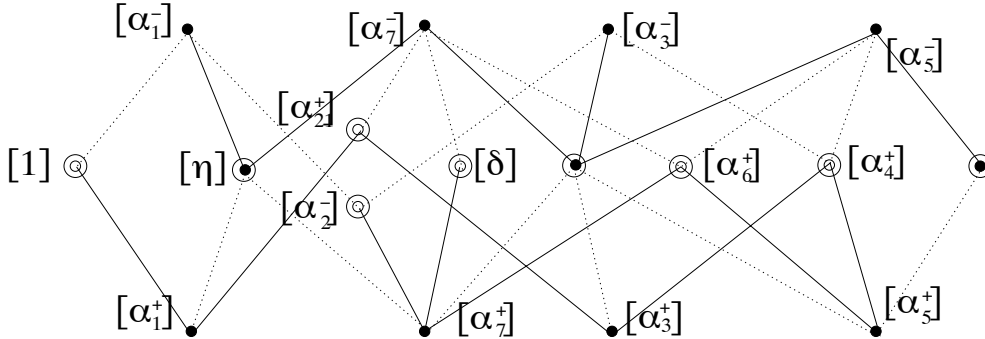


Figure 3. Full system for \mathbb{E}_7

Focus on the solid lines for now (the graphs describing multiplication by α_1^+). The copy of \mathbb{D}_{10} is clear, as it represents the nimrep of the neutral system. K -theoretically, of course it is ${}^{16+2}K_{\text{SU}_2}^1(\text{SO}(3))$. Note that the 6 even vertices of \mathbb{D}_{10} form the Verlinde ring ${}^{(1;0,8+1)}K_{\text{SO}_3}^1(\text{SO}(3))$ of the neutral system. The odd vertices are then the 4 spinors ${}^{(1;1,8+1)}K_{\text{SO}_3}^1(\text{SO}(3))$. The 3 even vertices of the \mathbb{E}_7 graph can be identified with the image under level-rank duality of the nimrep for the triality modular invariant of $\text{SO}(8)_3$, namely the K -group ${}^{3+6}K_{\text{Spin}_8}^0(\text{Spin}(8))$ where the adjoint action is twisted by triality.

The remaining question is how to recover the 4 remaining odd vertices of \mathbb{E}_7 . It is tempting to guess that these are 4 spinors for the twisted adjoint action of $\text{Spin}(8)$ on itself, but apparently there are no spinors at shifted level $3+6$. In some sense we already know these 4 odd vertices: they are the odd vertices of the dotted copy of \mathbb{D}_{10} . In any case, most of the \mathbb{E}_7 full system is clear.

A proper interpretation of the \mathbb{E}_7 full system, as a single K -group, would presumably involve $\mathrm{SO}(24)$ at level 1, but although there are many candidates, it isn't clear yet to us which is the correct one. It would certainly help to understand K -theoretically level-rank duality — this should be possible along the lines of [71], but may require understanding cosets or conformal embeddings from this framework.

7 Outlook and speculations

This paper is the second of a series devoted to deepening the connection between twisted equivariant K -homology and conformal field theory. This concluding section entertains some further possibilities.

Deep connections between K -theory and conformal field theory/string theory have been known for some time. But this paper establishes fundamental and systematic roles K -theory plays in a much more extensive range of CFT structures than had been previously appreciated. There is much more to do though.

One of the simplest examples of an orbifold is the *permutation orbifold*: let H be any subgroup of the symmetric group S_n , and let \mathcal{V} be any VOA (or RCFT); the permutation orbifold is $\mathcal{V}^{\otimes n}/H$, where H acts by permuting the copies of \mathcal{V} . A basic fact (see e.g. [2]) is that successive permutation orbifolds, first by $G < S_m$ and then by $H < S_n$, is equivalent to a single one of \mathcal{V} by the wreath product $G \wr H < S_{mn}$. Now, the modular data of the Drinfeld double of finite group G can be regarded as that of the permutation orbifold of a holomorphic VOA \mathcal{V} by G . Thus we obtain the observation that the Verlinde ring of the permutation orbifold (by any subgroup H of S_n) of the Drinfeld double of G , has the expression $K_{G \wr H}^0(G \wr H)$. This simple example is probably worth studying in more detail: see e.g. [67, 45] for some of the rich structure present. However the K -theoretic treatment of permutation orbifolds of loop group data is still not clear to us (see the examples given in [22]).

It would be very interesting to interpret K -theoretically the Goddard-Kent-Olive coset construction [46]. Of course *coset* here is not meant to be taken literally — e.g. it does not refer physically to a string living on the homogeneous space G/H . Algebraically, it involves commutants: e.g. the coset of a vertex operator algebra \mathcal{V} by a subalgebra \mathcal{W} is the commutant of \mathcal{W} in \mathcal{V} ; the coset of subfactor $N \subset M$ by subfactor $S \subset N$ is the subfactor $S' \cap N \subset S' \cap M$. A promising K -theoretic approach is based on [43]: understand the coset model G/H as an orbifold of $G \times H$ by the intersection $Z(G) \cap Z(H)$ of centres. The main difficulty seems to be the orbifold part. This approach won't always work: e.g. maverick cosets [19] such as conformal embeddings have *identifications* of primaries not merely given by the simple-currents $Z(G) \cap Z(H)$. In any case the study of coset models provides further motivation for developing the theory of orbifolds.

Note that the group of D-brane charges for the modular invariants of $\mathrm{SU}(2)$ is given by the centre of the corresponding A-D-E Lie group (recall the discussion at the end of section 2.3). This surprising fact has a simple K -theoretic interpretation using the trivial action of the maximal torus of each of those groups on itself. The

boundary states then are given by the simple roots, and the D-brane charges can then be recovered as the inner products with appropriate weights. It would be interesting to understand this elegant (though rather mysterious) description of the D-brane charges for $SU(2)$, from our framework.

As mentioned in section 2.3, it isn't obvious how to interpret (2.12) when G is finite. Suppose we extrapolate the K -theoretic treatment of the Verlinde D-brane charges for Lie groups G as given in section 2.4, to the finite group setting. In both settings the Verlinde ring is given by ${}^\tau K_0^{G^{\text{ad}}}(G)$; this suggests postulating that the charges for the finite group should also be given by the forgetful map $K_G^0(G) \rightarrow K^0(G) = \mathbb{Z}(G)$. This map sends the primary (g, χ) to $\dim(\chi)$ times the conjugacy class of g . Thus the image of the forgetful map is a free \mathbb{Z} -module of dimension equal to the class number of G . But by the argument given at the end of section 2.3, for any Verlinde nimrep all charges are uniquely determined from the charge of the vacuum, and thus the charge-group of the Verlinde nimrep will always be cyclic. This means that the assignment of charges here can be given by the forgetful map only when G has class number 1, i.e. only for the trivial case $G = 1$.

So the analogy between finite groups and Lie groups is not perfect in the context of D-brane charges. Nevertheless, one may hope that the Verlinde charge-group \mathcal{M}_N for G finite is a (cyclic) subgroup of $K^0(G) = \mathbb{Z}(G)$. This would imply that whatever we take 'dim(g, χ)' to be, (2.12) will be satisfied *exactly* (i.e. with $M = \infty$). There are $\dim(\text{Ver}(G))$ independent ways to do this, given by the assignments $q_{(g, \chi)} = S_{(g, \chi), (g', \chi')} / S_{(1, 1), (g', \chi')}$ for each fixed primary (g', χ') ; for only one choice of (g', χ') , namely $(1, 1)$, will these 'charges' be positive integers. Therefore it is tempting to suggest that *the correct choice of 'dim(g, χ)' in (2.12) for G a finite group, is the quantum-dimension $\|K_g\| \dim(\chi)$, where K_g denotes the conjugacy class of g . It would be interesting to compare these quantum-dimensions with the dimensions of Nahm's special spaces, for the fixed-point subVOAs \mathcal{V}^G of holomorphic VOAs \mathcal{V} .*

Langlands duality relates the groups $SU(n)/\mathbb{Z}_d$ and $SU(n)/\mathbb{Z}_{n/d}$ for example. Could this be manifested perhaps through a T-duality between the corresponding modular invariants, e.g. between the \mathbb{D} -series and \mathbb{A} -series of $SU(2)$?

The Verlinde ring and nimrep both come with preferred bases, in which the structure constants are nonnegative integers. This appears to be more evident in the K -theory language than in K -homology. To understand this point, consider the finite group case. In either case the double cosets arise automatically from the orbit analysis, but the appearance of representation rings of stabilisers would require Poincaré duality if we use K -homology. For K -homology, what would arise naturally presumably would be conjugacy classes of the stabilisers — this is the *permutation basis* of the Verlinde ring, in which $SL(2, \mathbb{Z})$ has a monomial representation. Maybe this is suggestive: from K^* here we get the preferred basis of Verlinde ring by primaries, but from K_* we get the permutation basis in which the modular group action is cleanest. The modular group representation is much cloudier in the primary = K^* basis, and the fusion structure is much cloudier in the permutation = K_* basis. Incidentally, there is no analogue of the permutation basis in general for the loop group

case.

Acknowledgement.

The authors thank the University of Alberta Mathematics Dept, Banff International Research Station, Cardiff School of Mathematics, Dublin Institute for Advanced Study, Erwin-Schrödinger Institute (Vienna), Swansea University Dept of Computer Science, and Universität Würzburg Institut für Mathematik for generous hospitality while researching this paper. Their research was supported in part by EPSRC GR/S80592/01, EU-NCG Research Training Network: MRTN-CT-2006 031962, DAAD (Prodi Chair), and NSERC. We thank Matthias Gaberdiel, Nigel Higson and Constantin Teleman for conversations.

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